From Heisenberg-Euler Lagrangian to the discovery of the Chromomagnetic Gluon Condensate

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1. Heisenberg-Euler Effective Lagrangian in QED
   Leipzig 1936

2. Light by Light Scattering at LHC

3. Effective Lagrangian in YM theory and QCD 1976

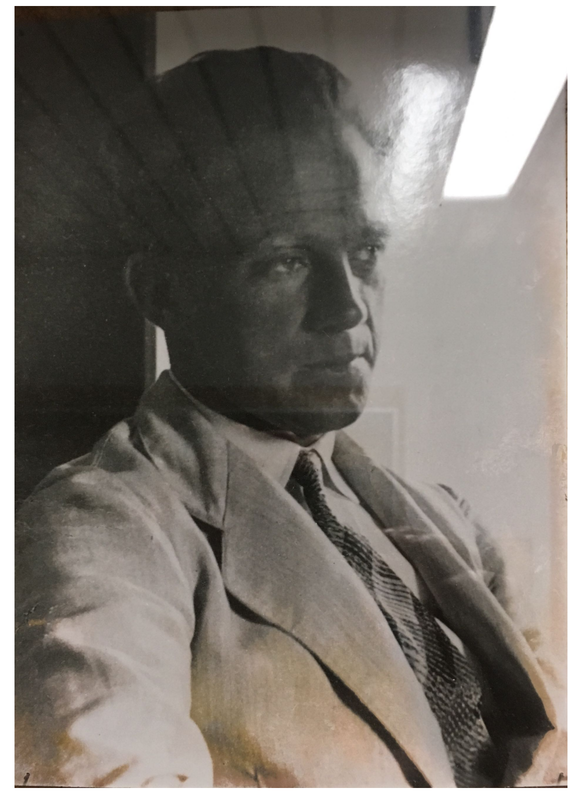
4. Chromomagnetic Gluon Condensation in QCD 1977
   Absence of the Imaginary part.

5. Absence of imaginary part in chromomagnetic field

6. Induced Effective Cosmological Constant

Based on lectures at the Leipzig University in occasion of the 80 Years of Heisenberg-Euler Lagrangian 1936-2016, ITP, Leipzig, November 21, 2016 and
40 Years of Discovery of the Chromomagnetic Gluon Condensation 1977-2017 at the Ludwig-Maximilian University München, Arnold Sommerfeld Colloquium at Center for Theoretical Physics, April 18, 2018.

Euler and Kockel 1935.
Heisenberg and Euler 1936



Hans Euler

Werner Heisenberg

Heisenberg-Euler Effective Lagrangian

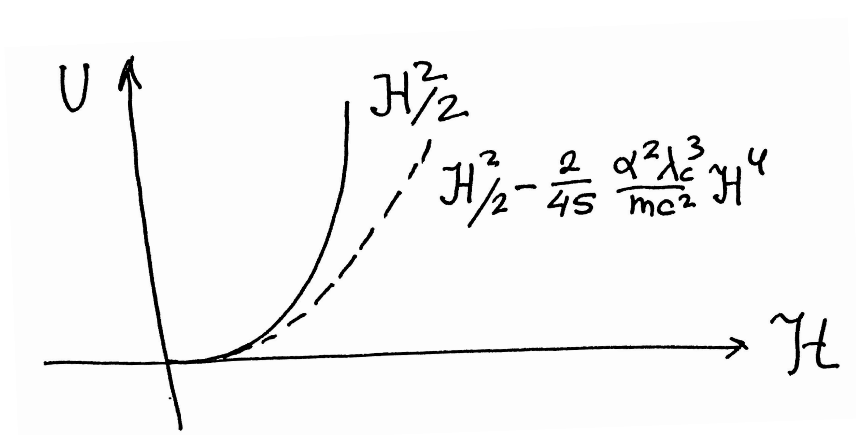
Performing the renormalisation one can get the effective Lagrangian

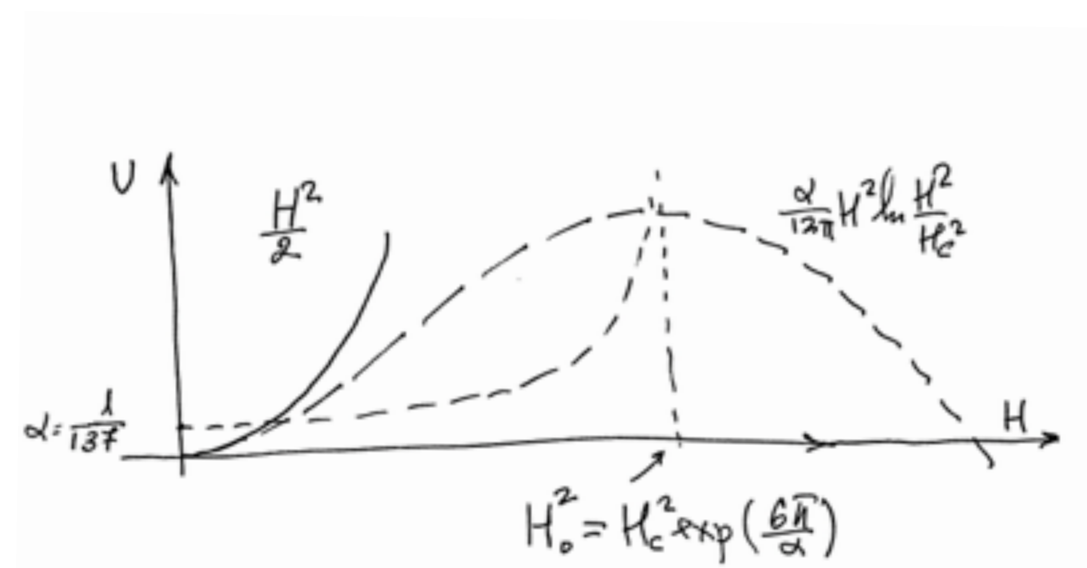
$$\mathcal{L}_{eff} = \frac{\mathcal{E}^2 - \mathcal{H}^2}{2} - 4\pi^2 mc^2 (\frac{mc}{h})^3 \int_0^\infty \frac{ds}{s^3} e^{-s} \{ \frac{as\cos(as)}{\sin(as)} \frac{bs\cosh(bs)}{\sinh(bs)} - 1 + \frac{a^2 - b^2}{3} s^2 \}$$

where

$$a = \frac{e\hbar\mathcal{E}}{m^2c^3}, \qquad b = \frac{e\hbar\mathcal{H}}{m^2c^3}$$

For strong magnetic fields a=0, b >> 1 the quantum correction term will have the form





$$\mathcal{L}_{eff} \approx -\frac{\mathcal{H}^2}{2} + \frac{2}{45}(\frac{e^2}{4\pi\hbar c})^2 (\frac{\hbar}{mc})^3 \frac{1}{mc^2} (\vec{\mathcal{H}}^2)^2$$

$$\mathcal{L}_{eff} \approx -\frac{\mathcal{H}^2}{2} + (\frac{e^2}{4\pi\hbar c}) \frac{\mathcal{H}^2}{12\pi} \ln(\frac{e\hbar\mathcal{H}}{m^2c^3})^2$$

Moscow zero

Heisenberg-Euler Effective Lagrangian

The renormalisation gives

$$\mathcal{L}_{eff} = \frac{\mathcal{E}^2 - \mathcal{H}^2}{2} - 4\pi^2 mc^2 (\frac{mc}{h})^3 \int_0^\infty \frac{ds}{s^3} e^{-s} \{ \frac{as\cos(as)}{\sin(as)} \ \frac{bs\cosh(bs)}{\sinh(bs)} - 1 + \frac{a^2 - b^2}{3} s^2 \}$$

where dimensionless fields are

$$a = \frac{e\hbar\mathcal{E}}{m^2c^3}, \qquad b = \frac{e\hbar\mathcal{H}}{m^2c^3}$$

$$mc^2 = 8.2 \cdot 10^{-7} \ \frac{g \ cm^2}{s^2} \qquad \lambda_c = \frac{\hbar}{mc} = 3.86 \cdot 10^{-11} cm \qquad \frac{mc^2}{(\frac{\hbar}{mc})^3} = 1.43 \cdot 10^{25} \frac{g}{cm \ s^2}$$

$$\mathcal{E}_c = 10^{16} \ Volt/cm \qquad U_{elec} = 0.8 \ 10^{26} \frac{g}{cm \ s^2}$$

$$\mathcal{H}_c = 4.4 \cdot 10^{13} \ Gauss \qquad U_{magnet} = 0.8 \cdot 10^{26} \frac{g}{cm \ s^2}$$

$$\mathcal{E}_b = 3 \cdot 10^4 \ Volt/cm \qquad U_{elec} = 4 \cdot 10^2 \frac{g}{cm \ s^2}$$ capacitor

$$\mathcal{H}_{neutron \ star} = 10^{15} \ Gauss \qquad U_{magnet} = 4 \cdot 10^{28} \frac{g}{cm \ s^2}$$

Euler-Kockel Effective Lagrangian

Euler and Kockel 1935.

$$\mathcal{L}_{eff} \approx \frac{\mathcal{E}^2 - \mathcal{H}^2}{2} + \frac{2}{45}(\frac{e^2}{4\pi\hbar c})^2(\frac{\hbar}{mc})^3\frac{1}{mc^2}\{(\vec{\mathcal{E}}^2 - \vec{\mathcal{H}}^2)^2 + 7(\vec{\mathcal{E}}\vec{\mathcal{H}})^2\}$$

$$D_i = \mathcal{E}_i + \frac{\partial \mathcal{L}_{vac}}{\partial \mathcal{E}_i}, \qquad B_i = \mathcal{H}_i - \frac{\partial \mathcal{L}_{vac}}{\partial \mathcal{H}_i}.$$

The electric displacement $\vec{D}$ and magnetic induction $\vec{B}$ induced in the vacuum are

$$D_i = \mathcal{E}_i + \frac{\partial \mathcal{L}_{vac}}{\partial \mathcal{E}_i} = \mathcal{E}_i + \frac{4}{45}(\frac{e^2}{4\pi\hbar c})^2(\frac{\hbar}{mc})^3\frac{1}{mc^2}\{2(\vec{\mathcal{E}}^2 - \vec{\mathcal{H}}^2)\mathcal{E}_i + 7(\vec{\mathcal{E}}\vec{\mathcal{H}})\mathcal{H}_i\}$$

$$, \qquad B_i = \mathcal{H}_i - \frac{\partial \mathcal{L}_{vac}}{\partial \mathcal{H}_i} = \mathcal{H}_i + \frac{4}{45}(\frac{e^2}{4\pi\hbar c})^2(\frac{\hbar}{mc})^3\frac{1}{mc^2}\{2(\vec{\mathcal{E}}^2 - \vec{\mathcal{H}}^2)\mathcal{H}_i - 7(\vec{\mathcal{E}}\vec{\mathcal{H}})\mathcal{E}_i\}.$$

For pure magnetic external field

$$B_i = \mathcal{H}_i(1 - \frac{8}{45}\frac{\alpha_{el}^2\lambda_c^3}{mc^2}\vec{\mathcal{H}}^2)$$

QED vacuum responds as diamagnet !

For pure electric external field

$$D_i = \mathcal{E}_i(1 + \frac{8}{45}\frac{\alpha_{el}^2\lambda_c^3}{mc^2}\vec{\mathcal{E}}^2)$$

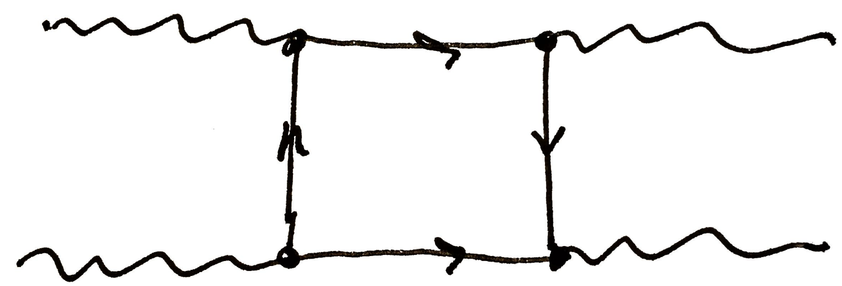
QED vacuum permittivity

Euler-Kockel Scattering of Light by Light

Euler and Kockel 1935.

For the week fields a << 1, b << 1 the first quantum correction term has the form

$$\mathcal{L}_{eff} \approx \frac{\mathcal{E}^2 - \mathcal{H}^2}{2} + \frac{2}{45}(\frac{e^2}{4\pi\hbar c})^2(\frac{\hbar}{mc})^3\frac{1}{mc^2}\{(\vec{\mathcal{E}}^2 - \vec{\mathcal{H}}^2)^2 + 7(\vec{\mathcal{E}}\vec{\mathcal{H}})^2\}$$

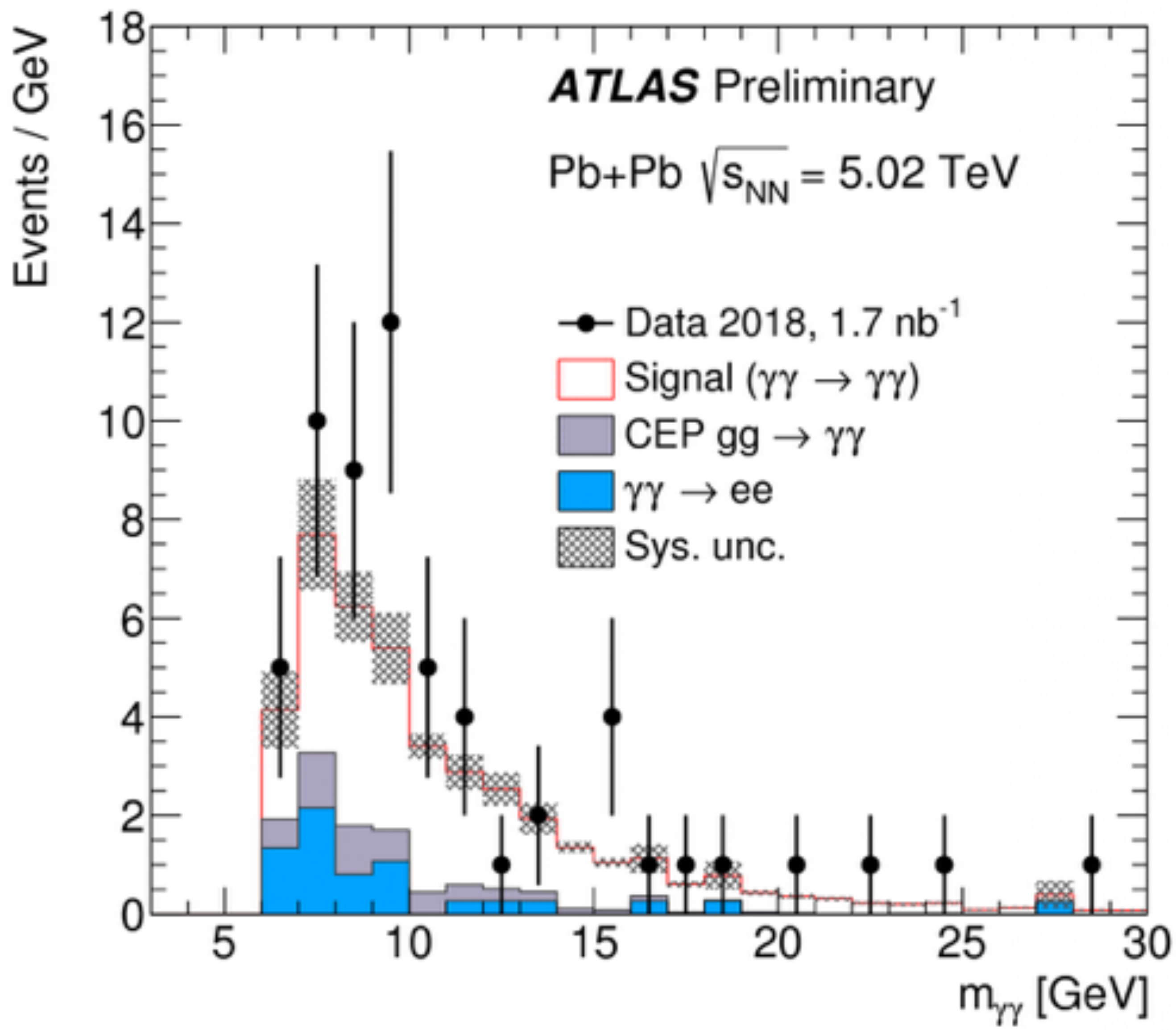


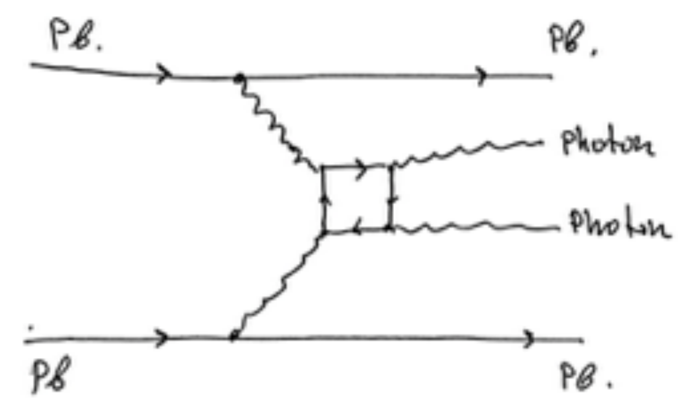
The light by light scattering cross section when $\omega \ll m$

$$\sigma_{\gamma\gamma\to\gamma\gamma} = \frac{1}{\pi}\frac{139}{90^2}\frac{56}{5}(\frac{e^2}{4\pi\hbar c})^4(\frac{\hbar}{mc})^2(\frac{\hbar\omega}{mc^2})^6 \approx 10^{-33}cm^2$$

Studying more than 4 billion events taken in 2015, the ATLAS collaboration found 13 candidates for light-by-light scattering. This result has a significance of 4.4 standard deviations, allowing the ATLAS collaboration to report the first direct evidence of this phenomenon.

$$\sigma_{\gamma\gamma\to\gamma\gamma} = 70 \pm 24(stat) \pm 17(syst)\ 10^{-33}cm^2$$





Mateusz Dyndal (CERN)

Invariant mass distribution of the measured final state photon pairs (markers), compared to the expected light-by-light scattering signal (red line) and expected background contributions (shaded areas). (Image: ATLAS Collaboration/CERN)

$$\sigma_{\gamma\gamma\to\gamma\gamma} = 78 \pm 13(stat) \pm 7(syst)\ 10^{-33} cm^2$$

$$E_T^{\gamma} > 3GeV \qquad |\eta_\gamma| < 2,37 \qquad m_{\gamma\gamma} > 6GeV$$

Sauter-Schwinger pair production

Sauter 1931
Schwinger 1951.

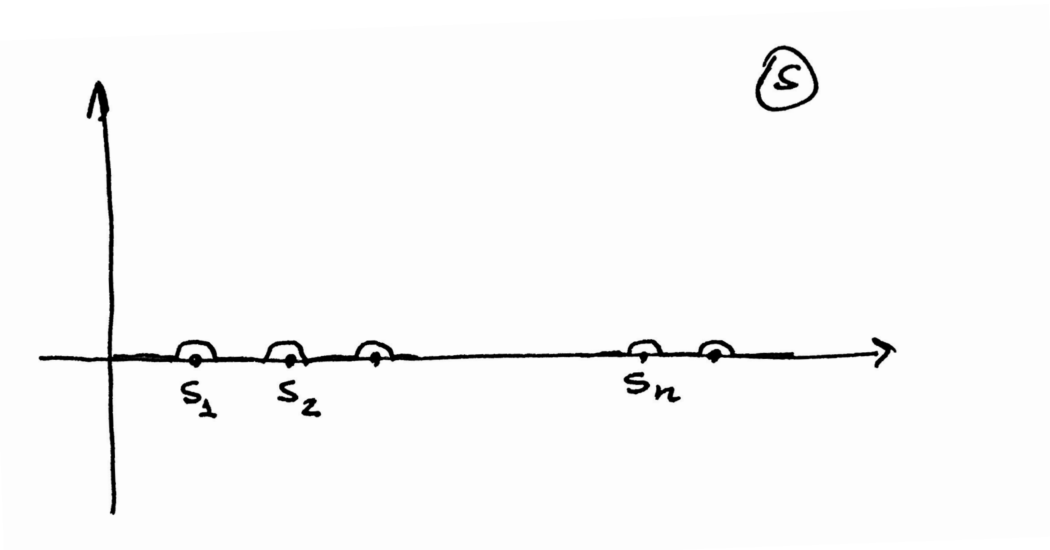
Considering pure electric field

$$\mathcal{L}^{(1)} = -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-m^2 s}\{\frac{(e\mathcal{E}s)\cos(e\mathcal{E}s)}{\sin(e\mathcal{E}s)} - 1 + \frac{1}{3}(e\mathcal{E}s)^2\}$$

it has singularities at $s_n = n\pi/e\mathcal{E}$ and the integration path is considered to lie above the real axis

$$2Im\mathcal{L}^{(1)} = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{s_n^2}e^{-m^2 s_n} = \frac{(e\mathcal{E})^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{m^2\pi}{e\mathcal{E}}n}$$

This is the probability, per unit time and per unit volume, that a electron-positron pair is created by the constant electric field.



Heisenberg-Euler Effective Lagrangian

1. The zeta function regularisation was introduced and used to express the Lagrangian
2. The renormalisation of Quantum Electrodynamics was clearly performed
3. The result is an infinite sum of the perturbation series
4. The asymptotic behaviour of QED at week and strong fields was derived
5. Weak field expansion coincides with the Euler-Kockel Effective Lagrangian
6. The tunnelling production of electron-positron pairs by strong electric field
7. The strong field behaviour demonstrate the vacuum instability, known as Moscow zero

Effective Lagrangian for Charged Vector Boson

Vanyashin and Terentev 1965
Duff and Ramon-Medrano 1975
Skalozub 1976
Bartalin, Matinyan and G.S. 1976

$$S_{YM}(A) = -\frac{1}{4}\int d^4x\ trG_{\mu\nu}G_{\mu\nu}, \qquad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\mu]$$

$$H_{\mu\nu}(\alpha) = \frac{\delta^2 S_{YM}(A)}{\delta A\ \delta A} = \eta_{\mu\nu}\nabla_\sigma(A)\nabla_\sigma(A) - 2gG_{\mu\nu} + (\alpha - 1)\nabla_\mu(A)\nabla_\nu(A),$$

$$H_{FP} = \nabla_\mu(A)\nabla_\mu(A).$$

Using proper time representation

$$\Gamma(A) = S_{YM}(A) - \frac{i}{2}\int_0^\infty \frac{ds}{s} Tre^{-iHs} + i\int_0^\infty \frac{ds}{s} Tre^{-iH_{FP}s}$$

or in equivalent form

$$\mathcal{L}_{eff} = \mathcal{L}_{YM} - \frac{i}{2}\int_0^\infty \frac{ds}{s} Tr(x|U(s)|x) + i\int_0^\infty \frac{ds}{s} Tr(x|U_0(s)|x)$$

where

$$U(s) = e^{-iHs}, \qquad U_0(s) = e^{-iH_{FP}s}$$

Effective Lagrangian in Quantum Chromodynamics

$$\mathcal{L}^{(1)} = -\frac{1}{32\pi^2}\int \frac{ds}{s^3} Trexp\{-L(s) + 2Ns\} + \frac{1}{16\pi^2}\int \frac{ds}{s^3} Trexp\{-L(s)\}$$

where the corresponding matrices are:

G.S. 1977
Matinyan and G.S. 1978

$$N = igG$$
$$K(s) = N\coth(Ns)$$
$$L(s) = \frac{1}{2} tr \ln[(Ns)\sinh(Ns)]$$

The effective Lagrangian take the following form

$$\mathcal{L}^{(1)} = -\frac{1}{8\pi^2}\int \frac{ds}{s^3} e^{-i\mu^2 s} \frac{(gF_1 s)\ (gF_2 s)}{\sinh(gF_1 s)\ \sinh(gF_2 s)} -$$
$$-\frac{1}{4\pi^2}\int \frac{ds}{s^3} e^{-i\mu^2 s} (gF_1 s)\ (gF_2 s)[\frac{\sinh(gF_1 s)}{\sinh(gF_2 s)} + \frac{\sinh(gF_2 s)}{\sinh(gF_1 s)}]$$

$$F_1^2 = -\mathcal{F} - (\mathcal{F}^2 + \mathcal{G}^2)^{1/2}, \qquad F_2^2 = -\mathcal{F} + (\mathcal{F}^2 + \mathcal{G}^2)^{1/2}$$

where we have introduced the infrared regularisation parameter μ

Effective Lagrangian in Quantum Chromodynamics

Renormalisation will give

$$\mathcal{L}^{(1)} = \quad \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}e^{-\mu^2 s}\Big(\frac{(gf_1 s)\ (gf_2 s)}{\sinh(gf_1 s)\ \sin(gf_2 s)} - 1 + \frac{1}{3}(gs)^2\mathcal{F}\Big) +$$
$$+ \frac{g^2}{4\pi^2}\int_0^\infty \frac{ds}{s}e^{-i\mu^2 s}\Big(f_1 f_2\frac{\sin(gf_1 s)}{\sinh(gf_2 s)} - f_1^2\Big)$$
$$- \frac{g^2}{4\pi^2}\int_0^\infty \frac{ds}{s}e^{-\mu^2 s}\Big(f_1 f_2\frac{\sin(gf_2 s)}{\sinh(gf_1 s)} - f_2^2\Big)$$

where s_0 is the ultraviolet cut off parameter and

$$f_1^2 = \mathcal{F} + (\mathcal{F}^2 + \mathcal{G}^2)^{1/2}, \qquad f_2^2 = -\mathcal{F} + (\mathcal{F}^2 + \mathcal{G}^2)^{1/2}$$

$$\mathcal{F} = \frac{1}{4}trG_{\mu\nu}G_{\mu\nu}, \qquad \mathcal{G} = \frac{1}{4}trG_{\mu\nu}G^*_{\mu\nu}.$$

Effective Lagrangian in Quantum Chromodynamics

First consider pure chromomagnetic case

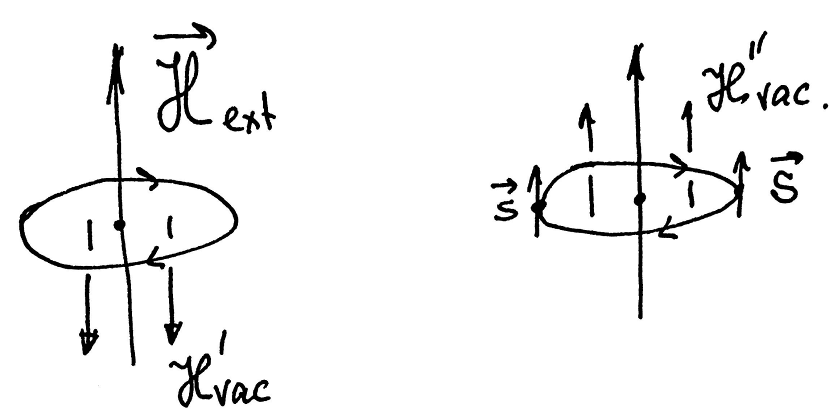
$$\mathcal{G} = 0, \mathcal{F} = (\mathcal{H}^2 - \mathcal{E}^2)/2 > 0, \qquad f_1^2 = 2\mathcal{F}, f_2^2 = 0$$

$$\mathcal{L}^{(1)} = + \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-\mu^2 s} \Big(\frac{g f_1 s}{\sinh(g f_1 s)} - 1 + \frac{1}{3}(gs)^2 \mathcal{F} \Big) + \\ + \frac{g^2}{4\pi^2} \int_0^\infty \frac{ds}{s} e^{-i\mu^2 s} \Big(\frac{f_1 \sin(g f_1 s)}{gs} - 2\mathcal{F} \Big)$$

The asymptotic behaviour for strong magnetic fields is

$$\mathcal{L}^{(1)} \approx \frac{(g\mathcal{H})^2}{48\pi^2} \ln \frac{g\mathcal{H}}{\mu^2} - \frac{(g\mathcal{H})^2}{4\pi^2} \ln \frac{g\mathcal{H}}{\mu^2} = - \frac{11}{48\pi^2} (g\mathcal{H})^2 \ln \frac{g\mathcal{H}}{\mu^2}$$

The *first term represent the diamagnetism* counteracting the external field caused by the charged gluons circulating in the vacuum due to the Lorentz force. The *second terms represent paramagnetism*, an effect associated with the polarisation of the gluon spin



G.S. 1977

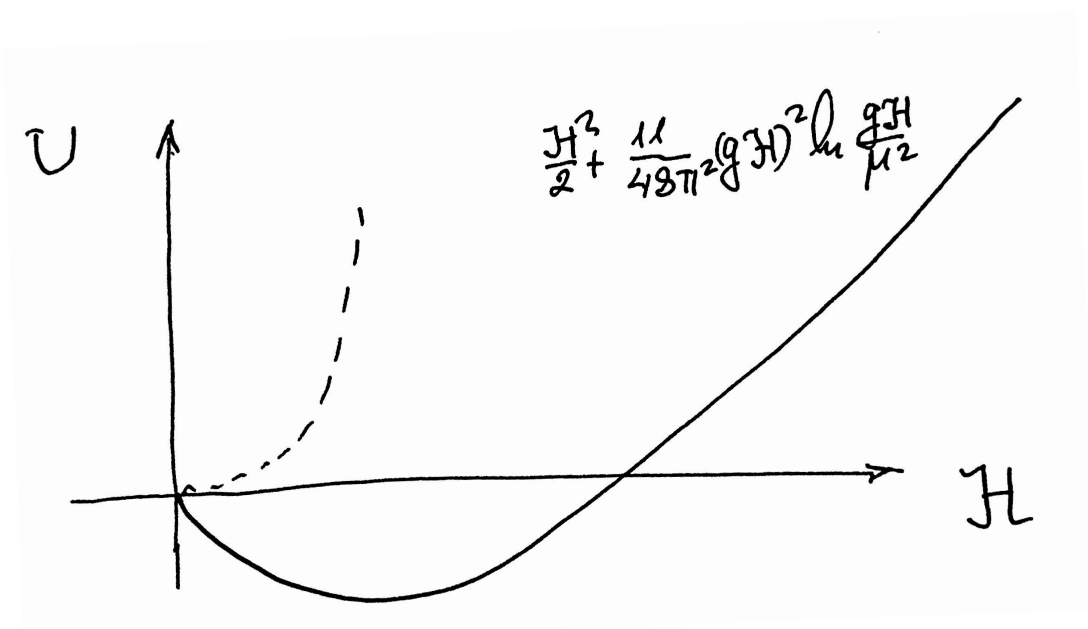
Effective Lagrangian in Quantum Chromodynamics

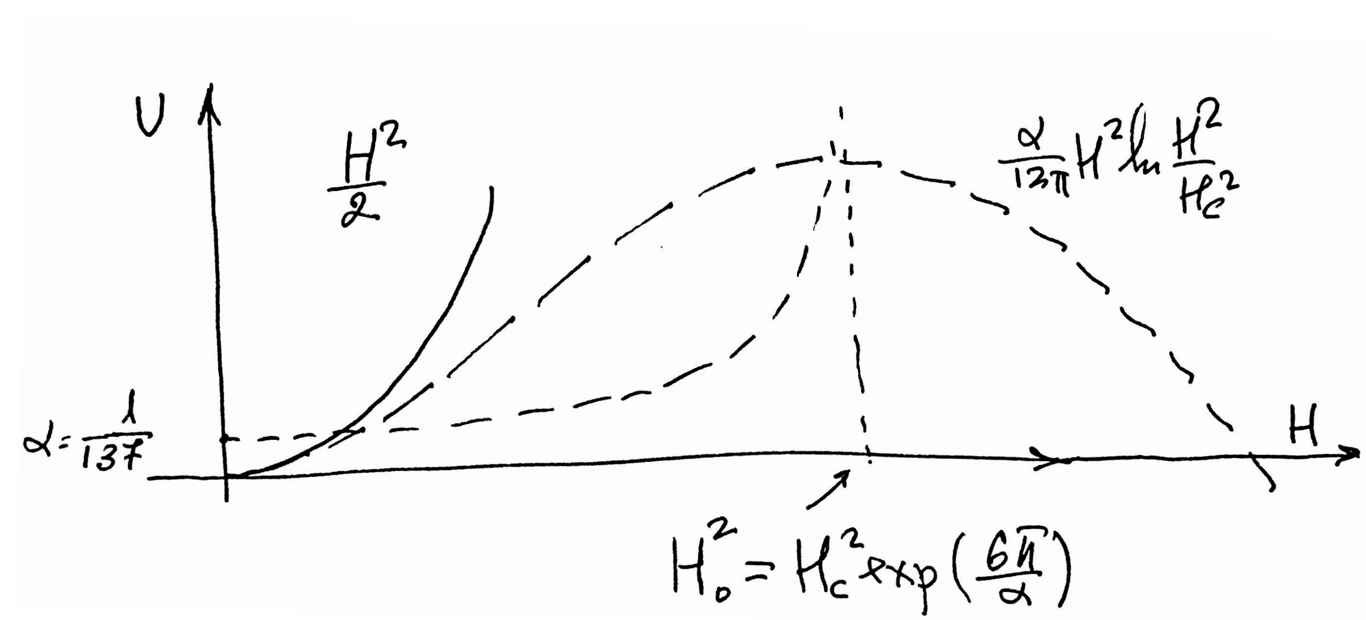
The asymptotic behaviour for strong magnetic fields is in QCD

$$\mathcal{L}^{(1)} \approx -\frac{11}{48\pi^2}(g\mathcal{H})^2 \ln \frac{g\mathcal{H}}{\mu^2}$$

As one can clearly see in QED the asymptotic is different

$$\mathcal{L}^{(1)} \approx +\frac{1}{24\pi^2}(e\mathcal{H})^2 \ln(\frac{e\mathcal{H}}{m^2})$$





The QCD vacuum is paramagnetic !

The QED vacuum is diamagnetic !

Renormalisation of Massless Theories

Loop expansion of the effective action has the following form

$$\Gamma = \sum_n \int dx_1 ... dx_n \Gamma^{(n) a_1 ... a_n}_{\mu_1 ... \mu_n}(x_1, ..., x_n) A^{a_1}_{\mu_1}(x_1) ... A^{a_n}_{\mu_n}(x_n)$$
$$= S_{YM} + W^{(1)} + W^{(2)} + ...$$

and can be expressed also as derivative expansion

$$\Gamma = \int d^4x [\bar{\mathcal{L}} + \tilde{\mathcal{L}} + \tilde{\tilde{\mathcal{L}}} + ...]$$

The $\bar{\mathcal{L}}$ depends on invariants $\mathcal{F}$ and $\mathcal{G}$, the $\tilde{\mathcal{L}}$ depends on first order covariant derivatives of $G_{\mu\nu}$ and so on. With the use of $\bar{\mathcal{L}}$ we can introduce the renormalisation of the effective action as

$$\frac{\partial \bar{\mathcal{L}}}{\partial \mathcal{F}}|_{t=\ln(\frac{2g^2\mathcal{F}}{\mu^4})=\mathcal{G}=0} = -1$$

allowing to define renormalisation

$$\Gamma_r = \Gamma_{un} + Z \ S_{YM}$$

Heisenberg-Euler Lagrangian in Massless Limit

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\Big|_{t=\frac{1}{2}\ln(\frac{2e^2|\mathcal{F}|}{\mu^4})=\mathcal{G}=0} = -1, \qquad \mathcal{L}^{(1)} = -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\frac{(ef_1 s)\cosh(ef_1 s)}{\sinh(ef_1 s)}$$

This leads to the following renormalisation of the Heisenberg-Euler Lagrangian in the massless limit

$$\mathcal{L}^{(1)} = -\frac{\mu^4}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\Big(\frac{as\cosh(as)}{\sinh(as)} - 1 - \frac{a^2 s}{2}\Big(\frac{\cosh s}{\sinh s} - \frac{s}{\sinh^2 s}\Big)\Big), \tag{2.34}$$

where

$$a^2 = 2e^2\mathcal{F}/\mu^4 = e^2\vec{\mathcal{H}}^2/\mu^4 \qquad , \quad \mathcal{G} = 0. \tag{2.35}$$

Lagrangian in massless QED is:

$$\mathcal{L}^{(1)}_{el} = \frac{e^2\mathcal{F}}{24\pi^2}\Big[\ln(\frac{2e^2\mathcal{F}}{\mu^4}) - 1\Big],$$

where $2\mathcal{F} = \vec{\mathcal{H}}^2 - \vec{\mathcal{E}}^2 > 0$, $\mathcal{G} = \vec{\mathcal{E}}\vec{\mathcal{H}} = 0$ and the effective Lagrangian will take the form

$$\mathcal{L} = -\mathcal{F} + \frac{e^2\mathcal{F}}{24\pi^2}\Big[\ln(\frac{2e^2\mathcal{F}}{\mu^4}) - 1\Big].$$

Chromomagnetic Gluon Condensate in QCD

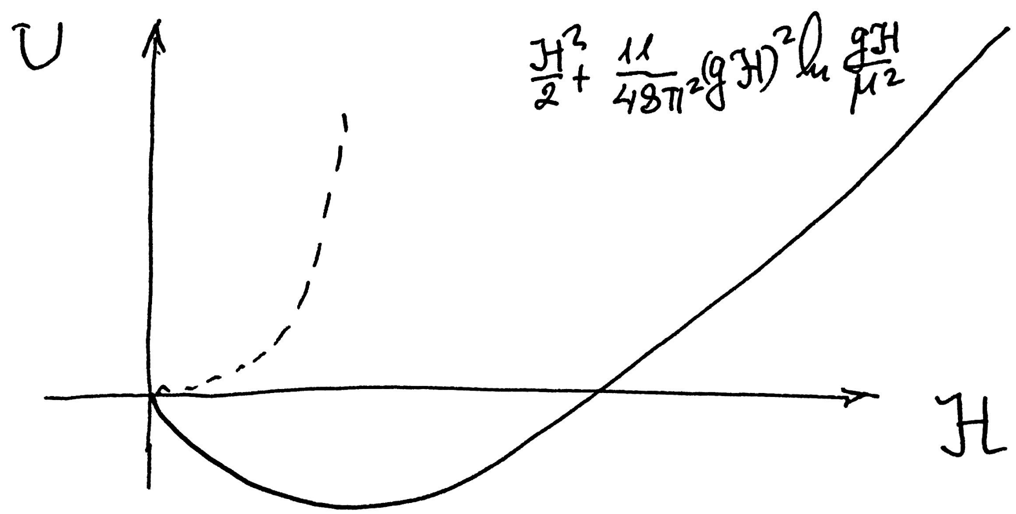
$$\mathcal{L}^{(1)} = \frac{\mu^4}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\Big(\frac{as}{\sinh as} - \frac{a^2 s}{2}\big(\frac{1}{\sinh s} - \frac{s\cosh s}{\sinh^2 s}\big)\Big) +$$
$$+\frac{\mu^4}{4\pi^2}\int_0^\infty \frac{ds}{s^3}\Big(as\sin(as) - \frac{a^2 s}{2}(\sin s + s\cos s)\Big)$$

where $a = g\mathcal{H}/\mu^2$. Taking the integrals one can get the energy density of the vacuum

$$U = \frac{\mathcal{H}^2}{2} + \frac{11}{48\pi^2}(g\mathcal{H})^2\Big(\ln\frac{g\mathcal{H}}{\mu^2} - \frac{1}{2}\Big) \equiv \mathcal{F} + \frac{11}{48\pi^2}g^2\mathcal{F}\Big(\ln\frac{2g^2\mathcal{F}}{\mu^4} - 1\Big)$$

where the invariant $\mathcal{F}$ is positive and correspond to the magnetic field configurations

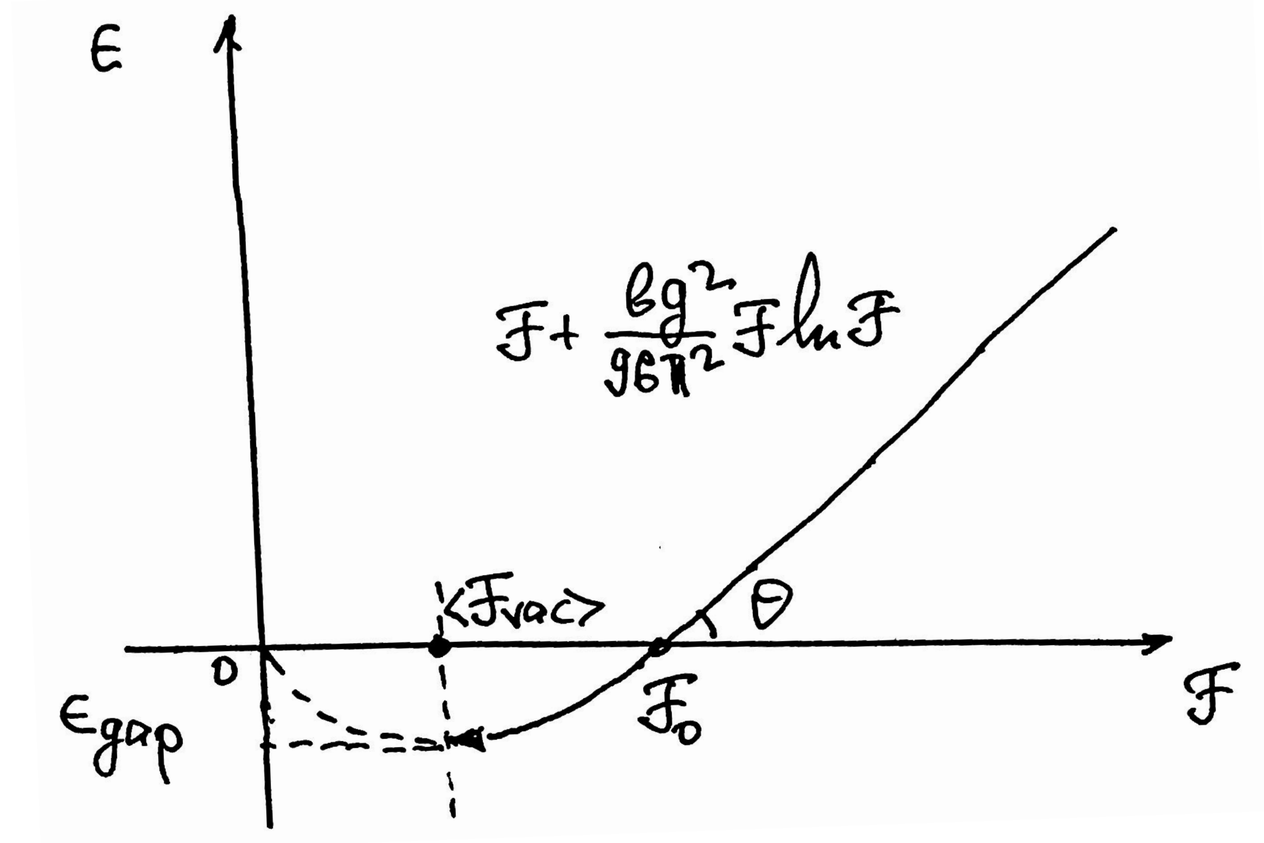
$$\mathcal{F} = \frac{1}{4}G^2_{\mu\nu} > 0, \quad \mathcal{G} = \frac{1}{4}G_{\mu\nu}G^*_{\mu\nu} = 0$$



$$< g^2 G^2_{\mu\nu} >_{vac} = 2\mu^4 \exp\big(-\frac{48\pi^2}{11g^2}\big)$$

G.S. Phys.Lett. B 71 1977

bag model interpretation H.Nielsen 1978



the vacuum energy density vanishes $\epsilon(\mathcal{F}_0) = 0$, as it is shown on Fig.

$$2g^2\mathcal{F}_0 = \mu^4 \exp\left(-\frac{96\pi^2}{11g^2N} + 1\right) = e < 2g^2\mathcal{F} >_{vac} .$$

The effective coupling constant at this field strength has the value

$$\bar{g}^2(\mathcal{F}_0) = \frac{96\pi^2}{11N} .$$

$$\bar{g}^2(\mathcal{F}_0) = \frac{96\pi^2}{11N} \ll 1 \quad \text{if} \quad N \gg \frac{96\pi^2}{11} . \qquad\qquad \tan\theta = \frac{11g^2N}{96\pi^2} \;>\; 0.$$

$$\mathcal{F}_n = e^{1-n} < \mathcal{F} >_{vac}, \qquad \bar{g}^2(\mathcal{F}_n) = \frac{96\pi^2}{11N(1-n)} \to 0$$

if the product $N(1-n) \to \infty$ is large and the t'Hooft coupling constant $g^2N = \lambda$ is fixed

Effective Lagrangian in massless QED

$$\mathcal{L}_e = -\mathcal{F} + \frac{e^2\mathcal{F}}{24\pi^2}\Big[\ln(\frac{2e^2\mathcal{F}}{\mu^4}) - 1\Big], \qquad \mathcal{F} = \frac{\vec{\mathcal{H}}^2 - \vec{\mathcal{E}}^2}{2}, \quad \mathcal{G} = \vec{\mathcal{E}}\vec{\mathcal{H}} = 0.$$

Effective Lagrangian in QCD

$$\mathcal{L}_q = -\mathcal{F} + \frac{N_f}{48\pi^2} g^2\mathcal{F}\Big[\ln(\frac{2g^2\mathcal{F}}{\mu^4}) - 1\Big] ,$$

where N_f is the number of quark flavours.

$$\mathcal{L}_g = -\mathcal{F} - \frac{11N}{96\pi^2} g^2\mathcal{F}\Big(\ln\frac{2g^2\mathcal{F}}{\mu^4} - 1\Big) , \qquad \mathcal{F} = \frac{\vec{\mathcal{H}}_a^2 - \vec{\mathcal{E}}_a^2}{2} > 0, \quad \mathcal{G} = \vec{\mathcal{E}}_a\vec{\mathcal{H}}_a = 0 .$$

Energy Momentum Tensors in QED and QCD

Schwinger 1951

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{F_{\mu\lambda}}F_{\nu\lambda} = -g_{\mu\nu}\mathcal{L} + F_{\mu\lambda}F_{\nu\lambda}\ \frac{\partial\mathcal{L}}{\partial\mathcal{F}} +\ g_{\mu\nu}\frac{\partial\mathcal{L}}{\partial\mathcal{G}}\mathcal{G}$$
$$= (F_{\mu\lambda}F_{\nu\lambda} - g_{\mu\nu}\frac{1}{4}F^2_{\lambda\rho})\frac{\partial\mathcal{L}}{\partial\mathcal{F}} - g_{\mu\nu}(\mathcal{L} - \mathcal{F}\frac{\partial\mathcal{L}}{\partial\mathcal{F}} - \mathcal{G}\frac{\partial\mathcal{L}}{\partial\mathcal{G}}).$$

$$T_{\mu\nu} = T^{el}_{\mu\nu}\Big[1 - \frac{e^2}{24\pi^2}\ln\frac{2e^2\mathcal{F}}{\mu^4}\Big] + g_{\mu\nu}\frac{e^2}{24\pi^2}\mathcal{F}, \qquad \mathcal{G} = 0,$$

G.S. 1977

$$T_{\mu\nu} = T^{YM}_{\mu\nu}\Big[1 + \frac{11Ng^2}{96\pi^2}\ln\frac{2g^2\mathcal{F}}{\mu^4}\Big] - g_{\mu\nu}\frac{11N}{96\pi^2}g^2\mathcal{F},$$

Renormalisation Group and Effective Action

$$\Gamma = \sum_n \int dx_1...dx_n \Gamma^{(n)a_1...a_n}_{\mu_1...\mu_n}(x_1,...,x_n) A^{a_1}_{\mu_1}(x_1)...A^{a_n}_{\mu_n}(x_n)$$

because the vertex functions and gauge fields transforms as follows

$$\Gamma^{(n)}_{r}\,{}^{a_1...a_n}_{\mu_1...\mu_n} = Z_3^{n/2}\Gamma^{(n)}_{un}\,{}^{a_1...a_n}_{\mu_1...\mu_n}, \qquad A^a_\mu(x)_r = Z_3^{-1}A^a_\mu(x)_{un}, \qquad g_r = Z_3^{1/2} g_{un}.$$

The renormalisation group equation takes the form

$$\{\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g)\frac{\partial}{\partial g} + \gamma(g)\int d^4x A^a_\mu(x)\frac{\delta}{\delta A^a_\mu(x)}\}\Gamma = 0$$

where $\beta(g)$ is the Callan-Symanzik beta function, the $\gamma(g)$ is the anomalous dimension. $\mathcal{G} = \vec{\mathcal{E}}\vec{\mathcal{H}} = 0$ it reduces to the form

$$\{\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g)\frac{\partial}{\partial g} + 2\gamma(g)\mathcal{F}\frac{\partial}{\partial \mathcal{F}}\}\mathcal{L} = 0,$$

Renormalisation Group and Effective Action

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = -\frac{g^2}{\bar{g}^2(t)}, \qquad \frac{d\bar{g}}{dt} = \beta(\bar{g}) \ , \qquad t = \frac{1}{2}\ln(2g^2\mathcal{F}/\mu^4).$$

The behaviour of the effective Lagrangian at strong fields is similar to the behaviour of the Gauge Field Theories at small distances or large momentum

We have

$$\vec{\mathcal{B}}_a = -\frac{\partial \mathcal{L}}{\partial \vec{\mathcal{H}}_a} \ = \ \mu_{vac} \ \vec{\mathcal{H}}_a.$$

The vacuum permeability

$$\mu_{QED} = 1 - \frac{e^2}{24\pi^2}\log(\frac{e^2\vec{\mathcal{H}}^2}{\mu^4}) \ < 1, \qquad \textit{diamagnetic},$$

$$\mu_{QCD} = 1 + \frac{g^2}{96\pi^2}(11N - 2N_f)\log\frac{g^2\vec{\mathcal{H}}_a^2}{\mu^4} \ > \ 1, \qquad \textit{paramagnetic}, \qquad N > \frac{2}{11}N_f.$$

Renormalisation Group and Effective Action

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = -\frac{g^2}{\bar{g}^2(t)}, \qquad \frac{d\bar{g}}{dt} = \beta(\bar{g}) \ , \qquad t = \frac{1}{2}\ln(2g^2\mathcal{F}/\mu^4).$$

then

$$\mathcal{L}(\mathcal{F}) = -\mu^4 \int \frac{e^{2t}}{\bar{g}^2(t)} dt \ , \qquad t = \frac{1}{2}\ln(2g^2\mathcal{F}/\mu^4)$$

the energy momentum tensor

$$T_{\mu\nu} = -\Big(G_{\mu\lambda}G_{\nu\lambda} - g_{\mu\nu}\frac{1}{4}G^2_{\lambda\rho}\Big)\frac{g^2}{\bar{g}^2(t)} + g_{\mu\nu}\Big(\int \frac{e^{2t}}{\bar{g}^2(t)}dt - \frac{1}{2}\frac{e^{2t}}{\bar{g}^2(t)}\Big)\mu^4$$

$$< 2g^2\mathcal{F} >_{vac} = \mu^4 \exp\left(-\frac{96\pi^2}{b\ g^2(\mu)}\right) = \Lambda^4_{QCD},$$

Higgs mode in Quantum Chromodynamics

The imaginary part of the effective Lagrangian is

G.S. 1977

$$Im\mathcal{L}^{(1)} = -\frac{gf_1}{4\pi^2}\int_0^\infty \frac{ds}{s^2}\sin(\mu^2 s)\sin(gf_1 s) + \frac{g^2 f_1^2}{4\pi^2}\int_0^\infty \frac{ds}{s}\sin(\mu^2 s)$$

$$= -\frac{gf_1}{4\pi^2}\frac{\pi}{2}gf_1 + \frac{g^2 f_1^2}{4\pi^2}\frac{\pi}{2} = -\frac{g^2 f_1^2}{8\pi} + \frac{g^2 f_1^2}{8\pi} = 0$$

where $f_1 = \sqrt{\vec{\mathcal{H}}^2} = \mathcal{H}$.

N.Nielsen and Olesen 1978
Ambjorn, N.Nielsen and Olesen 1979
H.Nielsen and Ninomia 1979
H.Nielsen and Olesen 1979
Ambjorn and Olesen1980

Deep understanding of the physics of unstable mode have been gained by the NBI group. As it was demonstrated in the series of NBI articles due to the unstable mode n=0

$$k_0^2 = k_{||}^2 + (2n+1\pm 2)gf_1$$

$$Im\mathcal{L}^{(1)} = Im\ \frac{gf_1}{4\pi^2}\int_{-\infty}^{\infty} dk_{||}\sqrt{k_{||}^2 - gf_1 - i\epsilon} = -\frac{g^2 f_1^2}{8\pi}$$

The dynamics of unstable mode may lead to the modulation of the vacuum field configurations.

Higgs mode in Quantum Chromodynamics

$$W_\mu = \frac{1}{\sqrt{2}}(A_\mu^1 + A_\mu^2),\ A_\mu = A_\mu^3 \text{ and } W = W_1 = -iW_2$$

N.Nielsen and Olesen 1978
Ambjorn, N.Nielsen and Olesen 1979
H.Nielsen and Ninomia 1979
H.Nielsen and Olesen 1979
Ambjorn and Olesen1980

$$W(x) = \int \frac{dk_2}{2\pi} e^{-\frac{1}{2}gH(x_1 - \frac{k_2}{gH})^2 + ik_2x_2}\Phi_{k_2}(x_0, x_3)$$

$$\frac{S_{higgs\ mode}}{\sqrt{2\pi}} = (gH)^{-1/2}\int \frac{dk_2}{2\pi}dx_0dx_3\Big(|\partial_\mu\Phi_{k_2}|^2 + gH|\Phi_{k_2}|^2 - \frac{1}{2}g^2\int\frac{dpdq}{(2\pi)^2}e^{-\frac{p^2+q^2}{2gH}}\Phi^*_{k_2+p}\Phi^*_{k_2+q}\Phi_{k_2}\Phi_{k_2+p+q}\Big)$$

where the $\Phi_{k_2}(x_0, x_3)$ is the dynamical variable of the higgs mode

C.A.Flory. *Covariant constant chromomagnetic fields and elimination of the one-loop instabilities,* SLAC-PUB-3244, October 1983.

Presenting the amplitude of the higgs mode and of the corresponding action in terms of alternative dimensionless variables $k_\mu \to k_\mu/\sqrt{gH}$, $x_\mu \to x_\mu\sqrt{gH}$ one can get[††]

$$(gH)^{-1/2}\ W = \int \frac{dk_2}{2\pi} e^{-\frac{1}{2}(x_1+k_2)^2} \Phi_{k_2}(x_0, x_3)$$

where $\Phi_{k_2}(x_0, x_3)$ is also dimensionless. The action of the higgs mode takes the form:

$$\frac{S_{higgs\ mode}}{\sqrt{2\pi}} = \int \frac{dk_2}{2\pi} dx_0 dx_3 \Big(|\partial_\mu \Phi_{k_2}|^2 + |\Phi_{k_2}|^2 - \frac{1}{2} g^2 \int \frac{dpdq}{(2\pi)^2} e^{-\frac{p^2+q^2}{2}} \Phi^*_{k_2+p} \Phi^*_{k_2+q} \Phi_{k_2} \Phi_{k_2+p+q} \Big).$$

What is essential in this representation is that the dependence on the chromomagnetic field does not show up in the Lagrangian and appears only in front of the higgs field amplitude $(gH)^{1/2}$

$$e^{-\mathcal{L}_{higgs\ mode}} \sim (gH)^{-\frac{1}{2}(\frac{gH}{2\pi})^2} \int \mathcal{D}\Phi_{k_2} e^{-S_{higgs\ mode}[\Phi_{k_2}]}$$

Contribution of the higgs mode is only through the integration measure

Effective Lagrangian in Quantum Chromodynamics

C.A.Flory. *Covariant constant chromomagnetic fields and elimination of the one-loop instabilities,* SLAC-PUB-3244, October 1983.

Further treatment of the unstable mode has been given by Flory. He demonstrated that a completely real energy density appears when one include the quartic terms of the Yang-Mills action for these modes. The argument of Flory is that the imaginary part of the effective action is an artefact of the one loop approximation.

One should use the complete Lagrangian for the unstable modes and not just the quadratic terms used in the one loop approximation. This calculation leads to the same expression for the real part of the Lagrangian and without imaginary part.

He demonstrated that a completely real energy density appears when one include the quartic terms of the Yang-Mills action for these modes.

$$\mathcal{L}^{(1)} = -(\frac{11}{96\pi^2} - \frac{1}{96\pi^2} + \frac{1}{8\pi^2})g^2H^2\Big(\ln(\frac{gH}{\mu^2} - \frac{1}{2}\Big)$$

$$k_0^2 = k_{||}^2 + (2n+1+2)gH, \quad n = 0,1,... \qquad k_0^2 = k_{||}^2 + (2n+1-2)gH, \quad n = 1,2,... \qquad k_0^2 = k_{||}^2 - gH \quad n = 0$$

Cosmological Constant

G.S. 1977

$$\mathcal{L} = -\mathcal{F} - \frac{11}{48\pi^2} g^2 \mathcal{F}\Big(\ln \frac{2g^2\mathcal{F}}{\mu^4} - 1 \Big)$$

$$T_{\mu\nu} = T^{YM}_{\mu\nu} \underline{\Big[1 + \frac{11g^2N}{96\pi^2} \ln \frac{2g^2\mathcal{F}}{\mu^4}\Big]} - g_{\mu\nu} \frac{11g^2N}{96\pi^2}\mathcal{F}, \qquad \mathcal{G} = 0.$$

$$< 2g^2\mathcal{F} >_{vac} = \mu^4 \exp\,(-\frac{96\pi^2}{11g^2N}).$$

at the vacuum state

$$< T_{\mu\nu} >_{vac} = -g_{\mu\nu}\ \frac{11}{96\pi^2} N < g^2\mathcal{F} >_{vac}\ .$$

$$\epsilon_{vac} = -P_{vac}$$

$$\epsilon_{vac} = \frac{c^4 \Lambda_{eff}}{8\pi G} \ = -\frac{11}{96\pi^2} N < g^2 \mathcal{F} >_{vac}$$

Zeldovich invariant equation

$$\varepsilon' = \frac{\varepsilon + \beta^2 P}{1 - \beta^2}, \qquad T'_{0x} = \frac{\beta\,(\varepsilon + P)}{\sqrt{1 - \beta^2}}, \qquad T'_{xx} = \frac{P + \beta^2 \varepsilon}{1 - \beta^2},$$

Cosmological Constant in Renormalisation group

$$T_{\mu\nu} = -\Big(G_{\mu\lambda}G_{\nu\lambda} - g_{\mu\nu}\frac{1}{4}G^2_{\lambda\rho}\Big)\frac{g^2}{\bar{g}^2(t)} + g_{\mu\nu}\Big(\int \frac{e^{2t}}{\bar{g}^2(t)}dt - \frac{1}{2}\frac{e^{2t}}{\bar{g}^2(t)}\Big)\mu^4$$

$$\frac{c^4}{8\pi G}\Lambda_{eff} \;=-\frac{b}{192\pi^2} < 2g^2\mathcal{F} >_{vac} \;=-\frac{b}{192\pi^2}\Lambda^4_{QCD}.$$

where $b \,=\, 11N \,-\, 2N_f$

$$< 2g^2\mathcal{F} >_{vac} = \mu^4 \exp\big(-\frac{96\pi^2}{b\; g^2(\mu)}\big) = \Lambda^4_{QCD},$$

Mirror QCD and Cosmological Constant

Chromomagnetic condensate in QCD

$$\epsilon_{vac} = \frac{c^4 \Lambda_{eff}}{8\pi G} = -\frac{11}{96\pi^2} N < g^2 \mathcal{F} >_{vac} \ , \qquad \mathcal{F} = \frac{\vec{\mathcal{H}}_a^{\ 2} - \vec{\mathcal{E}}_a^{\ 2}}{2} > 0.$$

Chromoelectric condensate in MQCD

$$\epsilon_{vac}^{MQCD} = -\frac{11}{96\pi^2} N < g^2 \mathcal{F}^{MQCD} >_{vac} \ , \qquad \mathcal{F}^{MQCD} = \frac{\vec{\mathcal{H}}_a^{\ 2} - \vec{\mathcal{E}}_a^{\ 2}}{2} < 0.$$

$$\epsilon = \epsilon_{vac}^{MQCD} + \epsilon_{vac}^{QCD} \approx 0.$$

R.Pasechnik et.al. arXiv:1804.09826

Thank You !