Gauged sigma model with Lie algebroid symmetry and moment map

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NI, Momentum sections in Hamiltonian mechanics and sigma models, arXiv:1905.02434 [math-ph].

§1. Introduction

Purpose

Physics

Geometry of physical theories with Lie algebroid symmetries

T-duality, U-duality, etc.

Cf. target space descriptions

Bessho, Carow-Watamura, Heller, Kaneko, NI, Watamura, '17,
Carow-Watamura, Kaneko, NI, Watamura '18

Math

The moment map theory is not sufficient to describe a symmetry and the Noether's theorem in physics.

Everything are categorified, groupoidified.

Plan of Talk

Nonlinear sigma models with B-field and gauging

Momentum sections

H-flux case

§2. 2D gauged nonlinear sigma model with B-field with boundary

Σ: a two dimensional manifold with boundary $\partial\Sigma \neq \emptyset$.
M: a d-dimensional target manifold.
$X : \Sigma \to M$ is a smooth map from Σ to M.

We start at

$$S = \frac{1}{2}\int_{\Sigma} g_{ij}(X) dX^i \wedge *dX^j + b_{ij}(X) dX^i \wedge dX^j,$$

where g is a metric and $b \in \Omega^2(M)$ is a closed 2-form on M.

Infinitesimal symmetries

We consider a symmetry on the target space described by a Killing vector ρ.

ρ has a structure of a Lie algebra $\mathfrak{g}$, precisely,
a map from $E = M \times \mathfrak{g}$, to a tangent bundle, $\rho : E \to TM$,

i.e. $\rho(e_a) = \rho_a^i(X)\partial_i$ by taking a basis e_a of E.

A Killing vector ρ defines an infinitesimal transformation of X as

$$\delta X^i \;=\; \rho(\epsilon)^i = \rho_a^i(X)\epsilon^a,$$

where $i = 1, 2, \cdots, d$ are indices of local coordinates on M, ϵ is a gauge parameter.

S is in invariant under this transformation iff

$$\begin{aligned}
\mathcal{L}_{\rho(e_a)} g &= 0, \\
\mathcal{L}_{\rho(e_a)} b &= d\beta_a, \\
[\rho(e_a), \rho(e_b)] &= \rho([e_a, e_b]),
\end{aligned}$$

where $\mathcal{L}$ is a Lie derivative and $\beta_a \in \Omega^1(M, E^*)$ is an arbitrary 1-form.

Generalization to a vector bundle

The above formula holds for a general vector bundle E.

Gauging

Hull, Spence '91

Chatzistavrakidis, Deser, Jonke '16, Chatzistavrakidis, Deser, Jonke, Strobl '17

We consider gauging of the previous symmetry. The transformation is gauged by introducing a connection 1-form $A \in \Omega^1(\Sigma, X^*E)$.

A pullback of a basis of a 1-form on M, dX^i, is 'gauged' as

$$F^i = DX^i = dX^i - \rho_a^i(X)A^a.$$

We assume A^a has a genuine infinitesimal gauge transformation,

$$\delta A^a \ = \ d\epsilon^a + [A, \epsilon]^a = d\epsilon^a + C_{bc}^a A^b \epsilon^c,$$

however, $C^a_{bc} = C^a_{bc}(X)$ is not necessarily constant but a local function on M.

Here, we consider a target space covariant gauge transformation by introducing (a pullback of) a target space connection on M, $\Gamma^a_{bi}(X)$:

$$\delta A^a = d\epsilon^a + C^a_{bc}(X) A^b \epsilon^c + \Gamma^a_{bi}(X) \epsilon^b D X^i.$$

In summary, we suppose gauge transformations,

$$\begin{aligned} \delta X^i &= \rho^i_a(X) \epsilon^a, \\ \delta A^a &= d\epsilon^a + C^a_{bc}(X) A^b \epsilon^c + \Gamma^a_{bi}(X) \epsilon^b D X^i. \end{aligned}$$

In order to make the action invariant under gauge symmetries, we take the following Hull-Spence type ansatz with boundary:

$$
\begin{aligned}
S &= \frac{1}{2}\int_{\Sigma} g_{ij}(X) DX^i \wedge *DX^j + b_{ij}(X) dX^i \wedge dX^j \\
&\quad + \int_{\partial\Sigma} \eta_i(X) dX^i + \mu_a(X) A^a,
\end{aligned}
$$

where η_i and μ_a are arbitrary functions of X.

Target space geometry

Metric

Requirement $\delta S = 0$ gives the following conditions for g and ρ,

$$\mathcal{L}_{\rho(e_a)} g = \Gamma_a^b \vee \iota_{\rho(e_b)} g,$$
$$[\rho(e_a), \rho(e_b)] = \rho([e_a, e_b]),$$

where $\vee$ is a symmetric product of 1-forms.

B-field term

Gauge invariance requirement for a B-field term with boundary terms imposes conditions,

$$\mu_a = -\eta_i \rho^i_a,$$
$$\rho^j_a b_{ji} + \rho^j_a \partial_j \eta_i + \eta_j \partial_i \rho^j_a + \Gamma^b{}_{ai} \mu_b = 0,$$
$$\rho^i_a \partial_i \mu_b - C^c_{ab} \mu_c - \rho^i_b \Gamma^c{}_{ai} \mu_c = 0,$$

What is this geometry?

§3. Lie algebroid and momentum section

Lie algebroid

Pradines '67

Let E be a vector bundle over a smooth manifold M. A *Lie algebroid* $(E, \rho, [-,-])$ is a vector bundle E with a bundle map $\rho : E \to TM$ and a Lie bracket $[-,-] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying the Leibniz rule,

$$[e_1, fe_2] \;=\; f[e_1, e_2] + \rho(e_1)f \cdot e_2,$$

where $e_i \in \Gamma(E)$ and $f \in C^\infty(M)$.

A bundle map ρ is called an anchor map.

Lie algebroid differential

$\Gamma(\wedge^\bullet E^*)$ is a space of exterior algebra on a Lie algebroid E. We define a Lie algebroid differential ${}^E d$ such that $({}^E d)^2 = 0$.

A *Lie algebroid differential* ${}^E d : \Gamma(\wedge^m E^*) \to \Gamma(\wedge^{m+1} E^*)$ for $\alpha \in \Gamma(\wedge^m E^*)$ and $e_i \in \Gamma(E)$ is defined by

$$\begin{aligned} {}^E d\alpha(e_1, \ldots, e_{m+1}) \;&=\; \sum_{i=1}^{m+1} (-1)^{i-1} \rho(e_i) \alpha(e_1, \ldots, \check{e}_i, \ldots, e_{m+1}) \\ &\quad + \sum_{i,j} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1}). \end{aligned}$$

Connection

Let $(E, \rho, [-,-])$ be a Lie algebroid over M.

A connection (a linear connection) on E is introduced as a covariant derivative $D : \Gamma(E) \to \Gamma(E \otimes T^*M)$. A connection is extended to $\Gamma(M, \wedge^\bullet T^*M \otimes E)$ as a degree 1 operator.

Pre-symplectic manifold

Let M be a smooth manifold. (M, B) is a pre-symplectic manifold if a 2-form $B \in \Omega^2(M)$ is closed, $dB = 0$.

Momentum section

Blohmann, Weinstein '18, Kotov, Strobl '16

$\gamma \in \Omega^1(M, E^*)$ is an E^*-valued 1-form defined by

$$\langle \gamma(v)\,, e \rangle = -B(v, \rho(e)),$$

for any $e \in \Gamma(E)$ and $v \in \mathcal{X}(M)$. Here $\langle - \,, - \rangle$ is a natural pairing of TM and T^*M.

In local coordinates, $\gamma_{ia} = -B_{ij}\rho_a^j$.

We introduce the following three conditions for a Lie algebroid E on a pre-symplectic manifold (M, B).

(H1) E is a *presymplectically anchored with respect to* D if

$$D\gamma = 0.$$

(H2) A section $\mu \in \Gamma(E^*)$ is a *D-momentum section* if

$$D\mu = \gamma.$$

(H3) A D-momentum section μ is *bracket-compatible* if for all sections $e_1, e_2 \in \Gamma(E)$,

$${}^{E}d\mu(e_1, e_2) = -\langle \gamma(\rho(e_1)) \, , e_2 \rangle.$$

A Lie algebroid E with D with (H1), (H2) and (H3) is called a *Hamiltonian*.

Lie algebra case: momentum map

Suppose B is nondegenerate, i.e., B is a symplectic form and $E = M \times \mathfrak{g}$. In this case, we can take a zero connection, $D = d$.

Then, (H2) and (H3) reduce to the definition of an infinitesimally equivariant *momentum map*,
$d\mu(e) = \iota_{\rho(e)} B$ and $\mathrm{ad}^*_{e_1} \mu(e_2) = \mu([e_1, e_2])$.

(H1) is trivial since $d^2 = 0$ in this case.

Structure of gauged linear sigma model NI '19

Three conditions,

$$
\begin{aligned}
&\mu_a = -\eta_i \rho_a^i, \\
&\rho_a^j b_{ji} + \rho_a^j \partial_j \eta_i + \eta_j \partial_i \rho_a^j + \Gamma^b_{\ ai} \mu_b = 0, \\
&\rho_a^i \partial_i \mu_b - C^c_{ab} \mu_c - \rho_b^i \Gamma^c_{\ ai} \mu_c = 0,
\end{aligned}
$$

are equivalent to (H2) and (H3), where $B = b + d\eta$.

i.e.In a gauged sigma model with boundary, $\mu = -\iota_\rho \eta \in \Gamma(E^*)$ is a bracket compatible D-momentum section, where $B = b + d\eta \in \Omega^2(M)$ and D is a connection defined by Γ.

Closure of gauge algebra

Requirement of closure of a gauge transformation of A, $[\delta_1, \delta_2] \sim \delta_3$ gives one more condition. An E-curvature is equal to zero.

(H1) is related to the condition. But they are not equivalent.

We need change the definition?

Cf. Bouwknegt, Bugden, Klimcik, Wright, '17, Wright '19

§4. Nonlinear sigma model with H-fluxes

n-dimensional NLSM

Ξ is an $n+1$-dimensional manifold with boundary $\Sigma = \partial\Xi$. We consider the following sigma model with Wess-Zumino term,

$$S = \int_\Sigma \frac{1}{2} g_{ij}(X) dX^i \wedge *dX^j + \int_\Xi X^* h,$$

$X^*h = \frac{1}{(n+1)!} h_{i_1 \cdots i_{n+1}}(X) dX^{i_1} \wedge \cdots \wedge dX^{i_{n+1}}$ is a pullback of a closed $n+1$-form h on M.

The $n=2$ case is a string theory with NS H-flux.

Gauging

Let E be a Lie algebroid as before.

We consider gauging with the same gauge transformations,

$$\begin{aligned} \delta X^i &= \rho_a^i(X)\epsilon^a, \\ \delta A^a &= d\epsilon^a + C_{bc}^a(X) A^b \epsilon^c + \Gamma_{bi}^a(X)\epsilon^b DX^i. \end{aligned}$$

We take a Hull-Spence type ansatz for a gauged action,

$$S = S_g + S_h + S_\eta,$$

where

$$
\begin{aligned}
S_g &= \int_\Sigma \frac{1}{2} g_{ij} DX^i \wedge *DX^j \\
S_h &= \int_\Xi \frac{1}{(n+1)!} h_{i_1 \cdots i_{n+1}}(X) dX^{i_1} \wedge \cdots \wedge dX^{i_{n+1}}, \\
S_\eta &= \int_\Sigma \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \eta^{(k)}_{i_1 \cdots i_k a_{k+1} \cdots a_n}(X) dX^{i_1} \wedge \cdots \wedge dX^{i_k} \\
& \quad \wedge A^{a_{k+1}} \wedge \cdots \wedge A^{a_n},
\end{aligned}
$$

where $\eta^{(k)}$ is a pullback of a k-form on M taking a value on $\wedge^{n-k} E^*$, i.e., $\eta^{(k)} \in X^* \Omega^k(M, \wedge^{n-k} E^*)$.

Geometric conditions

We impose $\delta S = 0$.

g_{ij}

The same as before,

$$\mathcal{L}_{\rho(e_a)} g = \Gamma_a^b \vee \iota_{\rho(e_b)} g.$$

h and $\eta^{(k)}$

$e, e_i \in \Gamma(E)$, $(i = k, \ldots, n)$, $\widetilde{h} = h + d\eta^{(n)}$, Γ is a connection 1-form on E, and $\langle - \, , - \rangle$ is a natural pairing of E^* and E. Notation $\overset{\wedge}{,}$ means both a wedge product on $\Omega^k(M)$ and a pairing of E and E^*.

We obtain the following conditions.
Two algebraic conditions,

$$\eta^{(k-1)}(e_k, \ldots, e_n) = (-1)^k \iota_{\rho(e_k)} \eta^{(k)}(e_{k+1}, \ldots, e_n) + \mathrm{Cycl}(e_k, \ldots, e_n), \tag{1}$$

$$\iota_{\rho(e_k)} \eta^{(k)}(e_{k+1}, \ldots, e_{k+m}, \ldots, e_n) + \iota_{\rho(e_{k+m})} \eta^{(k)}(e_{k+1}, \ldots, e_k, \ldots, e_n) = 0, \qquad (k = 1, \ldots, n-1, m = 1, \ldots, n-k) \tag{2}$$

and three differential equations,

$$D\eta^{(n-1)}(e) = \iota_{\rho(e)} \widetilde{h}, \qquad (k = n) \tag{3}$$

$$
\begin{aligned}
&\mathcal{L}_\rho(e)\eta^{(k)}(e_{k+1},\ldots,e_n) + \sum_{i=1}^{n-k}(-1)^i \eta^{(k)}([e,e_{k+i}],e_{k+1},\ldots,\check{e}_{k+i},\ldots,e_n) \\
&+ \sum_{i=1}^{n-k}(-1)^i \langle \Gamma, \rho(e)\rangle \wedge \eta^{(k)}(e_{k+1},\ldots,e_n) \\
&- \sum_{i=1}^{n-k}(-1)^i \Gamma(e) \wedge \iota_{\rho(e_{k+i})}\eta^{(k)}(e_{k+1},\ldots,\check{e}_{k+i},\ldots,e_n) \\
&+ \sum_{i=1}^{n-k}(-1)^i \langle \iota_{\rho(e_{k+i})}\Gamma(e) \overset{\wedge}{,} \eta^{(k)}(e_{k+1},\ldots,\check{e}_{k+i},\ldots,e_n)\rangle = 0, \qquad (k=1,\ldots,n-1)
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&\mathcal{L}_\rho(e)\eta^{(0)}(e_1,\ldots,e_n) + \sum_{i=1}^{n}(-1)^i \eta^{(0)}([e,e_{k+i}],e_{k+1},\ldots,\check{e}_{k+i},\ldots,e_n) \\
&+ \sum_{i=1}^{n}(-1)^i \langle \iota_{\rho(e_i)}\Gamma(e) \overset{\wedge}{,} \eta^{(0)}(e_1,\ldots,\check{e}_i,\ldots,e_n)\rangle = 0, \qquad (k=0)
\end{aligned}
\tag{5}
$$

Special cases

• In $n = 1$, Equations (1)–(5) reduce to conditions of the original momentum section (H2) and (H3) by setting $\mu = \eta^{(0)}$, $\gamma = \eta^{(1)}$ and $B = \widetilde{h}$.

• It is natural to impose the following condition corresponding to the condition (H1),

$$D\iota_\rho \widetilde{h} = 0. \tag{6}$$

• In $n = 2$, Equations (1)–(5) give gauging conditions of the target geometry in Chatzistavrakidis, Deser, Jonke and Strobl '16.

• For all n, Equations (1)–(5) are a generalization of a *momentum map (multimomentum map) on a multisymplectic manifold* with a Lie group action,

Carinena, Crampin, Ibort, '92, Gotay, Isenberg, Marsden, Montgomery, '97

by setting $\eta^{(k)} = 0$ for $k = 0, \ldots, n-2$. In this case, $\eta^{(n-1)}$ is a multimomentum map.

Momentum section on pre-n-plectic manifold NI '19

Let $(M, \widetilde{h})$ be a pre-n-plectic manifold, where $\widetilde{h}$ is a closed $n+1$-form, and $(E, \rho, [-,-])$ be a Lie algebroid over M. We define the following three conditions,

(HM1) E is a *pre-n-plectically anchored with respect to D* if Equation $D\iota_\rho \widetilde{h} = 0$, is satisfied.

(HM2) $\eta^{(n-1)} \in \Omega^{n-1}(M, E^*)$ is a *D-multimomentum (D-momentum) section* if it satisfies Equation (3),

$$D\eta^{(n-1)} = \iota_\rho \widetilde{h}.$$

(HM3) We define a descent set of multimomentum sections

$(\eta^{(k)})_{k=0}^{n-2}$ by Equations (1) and (2), where $\eta^{(k)} \in \Omega^k(M, \wedge^{n-k}E^*)$. A D-multimomentum section and its descents $(\eta^{(k)})_{k=0}^{n-1}$ are *bracket-compatible* if (4) and (5) are satisfied,

A Lie algebroid E with a connection D and a section $\eta^{(k)} \in \Omega^k(M, \wedge^{n-k}E^*)$, $k = 0, \dots, n-1$ is called $Hamiltonian$ if (HM1), (HM2) and (HM3) are satisfied.

We summarize a geometric structure of a gauge sigma model with a $n+1$-form flux h using the terminology of multimomentum sections.

We consider an n-dimensional gauged sigma model with WZ term, $\eta^{(k)} \in \Omega^k(M, \wedge^{n-k} E^*)$, $(k = 0, \ldots, n-1)$ are a bracket compatible D-multimomentum section and descents, with a pre-n-plectic form $\widetilde{h} = h + d\eta^{(n)}$.

§9. Conclusions

• We have shown that a two dimensional gauged sigma model with boundary has a momentum section and a Hamiltonian Lie algebroid structure.

• By generalizing it to a higher dimensional gauged sigma model with WZ term, target space geometry is described by the theory of a multimomentum section on a pre-multisymplectic manifold and a Hamiltonian Lie algebroid.

Outlook

- Quantization, localization formulas and equivariant cohomology
- Duality of string and M-theory
- Generalization to higher algebroids, such as a Courant algebroid.
- Comparison with multimoment map on mutlisymplectic manifold.

Madsen, Swann, '12, Callies, Fregier, Rogers and Zambon, '13, Herman, '18

Thank you for your attention!