Necessity and Insufficiency of Scale Invariance for solving Cosmological Constant Problem

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1 Cosmological Constant Problem

Dark Clouds hanging over the two well-established theories

Quantum Field Theory $\Longleftrightarrow$ Einstein Gravity Theory

I first explain my view point on WHAT IS actually the PROBLEM.

Presently observed Dark Energy Λ_0, looks like a small Cosmological Constant (CC):

$$\text{Present observed CC} \quad 10^{-29}\mathrm{gr/cm}^3 \sim 10^{-47}\mathrm{GeV}^4 \sim (1\,\mathrm{meV})^4 \equiv \Lambda_0 \quad (1)$$

I do NOT try to explain this tiny CC now, which will eventually be explained after our CC problem is solved. However, we use it as the scale unit Λ_0 of our discussion in the Introduction.

What is the true problem?

$\rightarrow$ Essential point: multiple mass scales are involved!

There are several dynamical symmetry breakings and they are necessarily accompanied by Vacuum Condensation Energy (potential energy):

In particular, from the success of the Standard Model, we are confident of the existence of at least TWO symmetry breakings:

$$\text{Higgs Condensation} \sim \quad (\ 200\,\text{GeV}\ )^4 \ \sim 10^9\text{GeV}^4 \sim 10^{56}\Lambda_0$$
$$\text{QCD Chiral Condensation } \langle\bar{q}q\rangle^{4/3} \sim \quad (\ 200\,\text{MeV}\ )^4 \sim 10^{-3}\text{GeV}^4 \sim 10^{44}\Lambda_0$$

Nevertheless, these seem not contributing to the Cosmological Constant!

It is a Super fine tuning problem:

$$c \ : \ \text{initially prepared CC } (>0)$$

$$c - 10^{56}\Lambda_0 \ : \ \text{should cancell, but leaving 1 part per } 10^{12}\text{; i.e., } \sim 10^{44}\Lambda_0$$

$$c - 10^{56}\Lambda_0 - 10^{44}\Lambda_0 \ : \ \text{should cancell, but leaving 1 part per } 10^{44}\text{; i.e., } \sim \Lambda_0$$

$$c - 10^{56}\Lambda_0 - 10^{44}\Lambda_0 \sim \Lambda_0 \ : \text{present Dark Energy}$$

$$c = \text{initially prepared CC}$$

$$\underbrace{123456,789012}_{12\text{ digits}}3456,7890123456,7890123456,7890123456,7890123456 \times \Lambda_0 \ \sim 10^{56}\Lambda_0$$

$$c + V_{\text{Higgs}} =$$

$$\underbrace{1234,5678901234,5678901234,5678901234,5678901234}_{44\text{ digits}} \times \Lambda_0 \ \sim 10^{44}\Lambda_0$$

$$c + V_{\text{Higgs}} + V_{\text{QCD}} = \text{present Dark Energy}$$

$$1 \times \Lambda_0 \ \sim \Lambda_0$$

Note that the vacuum energy is almost TOTALLY CANCELLED at each stage of spontaneous symmetry breaking as far as the the relevant energy scale order.

Contents

Part I: Scale Invariance is Necessary

2 Vacuum Energy $\simeq$ Potential energy

People may suspect: there are "TWO" origins of Cosmological Constant

$\boxed{\text{(Quantum) Vacuum Energy}}$

$$\sum_{\boldsymbol{k},s} \frac{1}{2}\hbar\omega_{\boldsymbol{k}} - \sum_{\boldsymbol{k},s} \hbar E_{\boldsymbol{k}} \tag{2}$$

Infinite, No controle, simply discarded

$\updownarrow$

$\boxed{\text{(Classical) Potential Energy}}$

$$V(\phi_c) : \text{potential} \tag{3}$$

Finite, vacuum condensation energy

They are separately stored in our (or my, at least) memory, but actually, almost the same object, as we see now.

We now show for the vacuum energies in the SM that

$$\text{quantum Vacuum Energy} = \text{Higgs Potential Energy} \tag{4}$$

Let us see this more explicitly. For that purpose, consider
Simplified SM:

$$\mathcal{L}_{\rm r} = \bar{\psi}\big(i\gamma^\mu\partial_\mu - y\phi(x)\big)\psi(x) + \frac{1}{2}\big(\partial^\mu\phi(x)\partial_\mu\phi(x) - m^2\phi^2(x)\big) - \frac{\lambda}{4!}\phi^4(x) - hm^4.$$

Effective Action (Effective Potential) is calculated prior to the vacuum choice.
(i.e., calculable independently of the choice of the vacuum)

1-loop effective potential in the Simplified SM

Use dimensional regularization for doing Mass-Independent (MI) renormalization

$$V(\phi, m^2) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + hm^4 + V_{\text{1-loop}} + \delta V^{(1)}_{\text{counterterms}}$$

$$V_{\text{1-loop}} = \frac{1}{2}\int\frac{d^4k}{i(2\pi)^4}\ln(-k^2 + \underbrace{m^2 + \frac{1}{2}\lambda\phi^2}_{=M_\phi^2(\phi)}) - 2\int\frac{d^4p}{i(2\pi)^4}\ln(-p^2 + \underbrace{y^2\phi^2}_{=M_\psi^2(\phi)})$$

Using dimensional formula

$$\frac{1}{2}\mu^{4-n}\int\frac{d^n k}{i(2\pi)^n}\ln(-k^2+M^2)=\frac{M^4}{64\pi^2}\Big(-\frac{1}{\bar{\varepsilon}}\underbrace{+\ln\frac{M^2}{\mu^2}-\frac{3}{2}}_{\text{Coleman-Weinberg potential}}\Big). \tag{5}$$

and dropping the $1/\bar{\varepsilon}$ parts in $\overline{\mathrm{MS}}$ renormalization scheme $\left(\frac{1}{\bar{\varepsilon}}=\frac{1}{\varepsilon}-\gamma+\ln 4\pi,\ \varepsilon=2-\frac{n}{2}\right)$, we get finite well-known renormalized 1-loop effective potential:

$$\begin{aligned}V(\phi,m^2)&=\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4+hm^4\\&+\frac{(m^2+\frac{1}{2}\lambda\phi^2)^2}{64\pi^2}\left(\ln\frac{m^2+\frac{1}{2}\lambda\phi^2}{\mu^2}-\frac{3}{2}\right)-4\frac{(y\phi)^4}{64\pi^2}\left(\ln\frac{y^2\phi^2}{\mu^2}-\frac{3}{2}\right)\end{aligned} \tag{6}$$

The divergences:

$$\begin{aligned}M_\phi^4(\phi)&=\left(m^2+\frac{\lambda}{2}\phi^2\right)^2=m^4+\lambda m^2\phi^2+\frac{\lambda^2}{4}\phi^4\\M_\psi^4(\phi)&=\left(y\phi\right)^4=y^4\,\phi^4\end{aligned} \tag{7}$$

These divergences are renormalized into λ and m^2, and h; The main part of quantum vacuum energies are already included in the classical potential $V(\phi)$.

3 Conclusions from these simple observation

As far as the matter fields and gauge fields are concerned, whose mass comes solely from the Higgs condensation $\langle\phi\rangle$,

> Their vacuum energies are calculable and finite quantities
> in terms of the renormalized λ parameters!

Note that this is because their divergences are proportional to ϕ^4. (At 1-loop, only ϕ^4 divergences appear.)

However, the Higgs itself is an exception! The divergences of the Higgs vacuum energy are not only $m^2\phi^2$ and ϕ^4 but also the zero-point function proprtional to m^4. In order to cancel that part, we have to prepare the counterterm:

$$h_0 m_0^4 = Z_h Z_m^2\, h m^4 = (1+F) h m^4$$
$$F^{(1)} h = \frac{1}{64\pi^2}\frac{1}{\bar{\varepsilon}}.$$

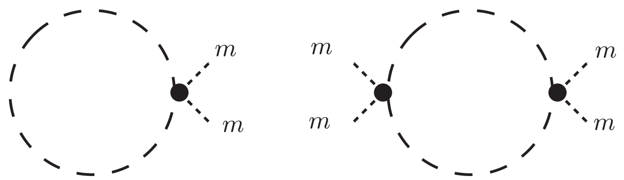

Figure 1: $\propto m^4$

And the renormalized CC term hm^4 is a Free Parameter. Then, there is no chance to explain CC.

Thus, for the calculability of CC, we need $m^2 = 0$, or

> No dimensionful parameters in the theory $\Rightarrow$ (Classical) Scale-Invariance

Part II: Scale Invariance is a Sufficient Condition?

4 Scale Invariance may solve the problem

Our world is almost scale invariant: that is, the standard model Lagrangian is scale invariant except for the Higgs mass term!

If the Higgs mass term comes from the spontaneous breaking of scale invariance at higher energy scale physics, the total system can be really be scale invariant:

$$\lambda(h^\dagger h - m^2)^2 \quad \rightarrow \quad \lambda(h^\dagger h - \varepsilon\Phi^2)^2. \tag{8}$$

where Φ may be a field which appear also in front of Einstein-Hilbert term:

$$\int d^4x\sqrt{-g}\;\Phi^2\,R \tag{9}$$

This idea is proposed by many authors including

M. Shaposhnikov and D. Zenhausern, Phys. Lett. B **671** (2009) 162

I. Antoniadis and N. C. Tsamis, Phys. Lett. **144B** (1984) 55.

E. T. Tomboulis, Nucl. Phys. B **329** (1990) 410.

C. Wetterich, Nucl. Phys. B **302** (1988) 668

But I will mainly follow Shaposhnikov and Zenhausern.

Antoniadis-Tsamis and Tomboulis papers appear very early and actually contains almost all basic ideas in this direction for solving the CC problem. Nevertheless those work use local SI which I think has to be useless:

$$\boxed{\text{Local SI theory with dilaton (without Weyl gauge field) is meaningless}}$$

($\because$) If the dilaton field $\Phi_0(x)$ is present, any system can be cast into local SI form: Indeed, by switching to scale invariant variables

$$\phi_i \;\rightarrow\; \Phi_0^{-d_i}\phi_i =: \phi_i' \tag{10}$$

with suitable power d_i depending on each field. And, the system reduces to the original Lagrangian in the unitary gauge $\Phi_0(x) = 1$. q.e.d.

Essentially the same but more detailed discussion is made in
N. C. Tsamis and R. P. Woodard, Annals Phys. **168** (1986) 457.

From now on, we always mean Global SI.

4.1 Classical Scale Invariance : wishful scenario

Suppose that our world has no dimensionful parameters.
Suppose that the effective potential V of the total system looks like

$$\begin{array}{ccccccc} V(\phi) = & V_0(\Phi) & + & V_1(\Phi, h) & + & V_2(\Phi, h, \varphi) \\ & \downarrow & & \downarrow & & \downarrow \\ & M & \gg & \mu & \gg & m \end{array}$$

and it is scale invariant. Then, classically, it satisfies the scale invariance relation :

$$\sum_i \phi^i \frac{\partial}{\partial \phi^i} V(\phi) = 4V(\phi), \tag{11}$$

so that the vacuum energy vanishes at any stationary point $\langle \phi^i \rangle = \phi_0^i$:

$$V(\phi_0) = 0.$$

Important point is that this holds at every stages of spontaneous symmetry breaking.

This miracle is realized since the scale invariance holds at each energy scale of spontaneous symmetry breaking.

For example, we can write a toy model of potentials.

$$\boxed{V_0(\Phi) = \tfrac{1}{2}\lambda_0(\Phi_1^2 - \varepsilon_0\Phi_0^2)^2,}$$

in terms of two real scalars Φ_0, Φ_1, to realize a VEV

$$\langle\Phi_0\rangle = M \quad \text{and} \quad \langle\Phi_1\rangle = \sqrt{\varepsilon_0}M \equiv M_1. \tag{12}$$

This M is totally spontaneous and we suppose it be Planck mass giving the Newton coupling constant via the scale invariant Einstein-Hilbert term

$$S_{\text{eff}} = \int d^4x\,\sqrt{-g}\Big\{c_1\Phi_0^2\,R + c_2R^2 + c_3R_{\mu\nu}R^{\mu\nu} + \cdots\Big\}$$

If GUT stage exists, ε_0 may be a constant as small as 10^{-4} and then Φ_1 gives the scalar field breaking GUT symmetry; e.g., $\Phi_1 : \mathbf{24}$ causing $SU(5) \to SU(3) \times SU(2) \times U(1)$.

$V_1(\Phi, h)$ part causes the electroweak symmetry breaking:

$$\boxed{V_1(\Phi, h) = \tfrac{1}{2}\lambda_1\left(h^\dagger h - \varepsilon_1\Phi_1^2\right)^2,}$$

with very small parameter $\varepsilon_1 \simeq 10^{-28}$. This reproduces the Higgs potential when h is the Higgs doublet field and $\varepsilon_1\Phi_1^2$ term is replaced by the VEV $\varepsilon_1 M_1^2 = \mu^2/\lambda_1 \sim (10^2\text{GeV})^2$.

$V_2(\Phi, h, \varphi)$ part causes the chiral symmetry breaking, e.g., $\mathrm{SU(2)_L}\times\mathrm{SU(2)_R} \to \mathrm{SU(2)_V}$. Using the 2×2 matrix scalar field $\varphi = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}$ (chiral sigma-model field), we may similarly write the potential

$$\boxed{V_2(\Phi, h, \varphi) = \tfrac{1}{4}\lambda_2 \left(\mathrm{tr}(\varphi^\dagger \varphi) - \varepsilon_2 \Phi_1^2\right)^2 + V_{\mathrm{break}}(\Phi, h, \varphi)}$$

with another small parameter $\varepsilon_2 \simeq 10^{-34}$. The first term reproduces the linear σ-model potential invariant under the chiral $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ transformation $\varphi \to g_\mathrm{L} \varphi g_\mathrm{R}$ when $\varepsilon_2 \Phi_1^2$ is replaced by the VEV $\varepsilon_2 M_1^2 = m^2/\lambda_2$. The last term V_{break} stands for the chiral symmetry breaking term which is caused by the explicit quark mass terms appearing as the result of tiny Yukawa couplings of u, d quarks, y_u, y_d, to the Higgs doublet h; e.g.,

$$V_{\mathrm{break}}(\Phi, h, \varphi) = \frac{1}{2} \varepsilon_2 \Phi_1^2 \, \mathrm{tr} \left(\varphi^\dagger \left(y_u \epsilon h^* \quad y_d h\right) + \mathrm{h.c.}\right)$$

4.2 Quantum Mechanically

Is there Anomaly for the Scale Invariance?

Usual answer is YES in quantum field theory. If we take account of the renormalization point μ, so that we have dimension counting identity

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_i \phi_i\frac{\partial}{\partial\phi_i}\right)V(\phi)=4V(\phi).$$

The anomaly $\mu(\partial/\partial\mu)V$ term may be replaced by RGE:

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_a \beta_a(\lambda)\frac{\partial}{\partial\lambda_a}+\sum_i \gamma_i(\lambda)\phi_i\frac{\partial}{\partial\phi_i}\right)V(\phi)=0$$

Shaposhnikov-Zenhausern's New Idea is: **SI exists even quantum mechanically**.

Quantum Scale Invariance

- Englert-Truffin-Gastmans, Nuc. Phys. B177(1976)407.
- M. Shaposhnikov and D. Zenhausern, Phys. Lett. B **671** (2009) 162
- See also Antoniadis-Tsamis, and , Tomboulis

Extension to n-dimension keeping S.I. is possible by introducing dilaton field Φ $\rightarrow$ **NO ANOMALY**.

1. **Usual dimensional regularization**

$$\begin{array}{rcll} \lambda\,(h^\dagger(x)h(x))^2 & \to & \lambda\,\mu^{4-n}(h^\dagger(x)h(x))^2 & [h]=\dfrac{n-2}{2} \\ y\,\bar{\psi}(x)\psi(x)h(x) & \to & y\,\mu^{\frac{4-n}{2}}\bar{\psi}(x)\psi(x)h(x) & [\psi]=\dfrac{n-1}{2} \end{array} \tag{13}$$

2. **SI prescription** Using 'dilaton' field $\Phi(x)$, $\mu \to \Phi^{\frac{2}{n-2}}$

$$\begin{array}{rcl} \lambda\,(h^\dagger(x)h(x))^2 & \to & \lambda\,[\Phi(x)^2]^{\frac{4-n}{n-2}}\,(h^\dagger(x)h(x))^2 \\ y\,\bar{\psi}(x)\psi(x)h(x) & \to & y\,[\Phi(x)]^{\frac{4-n}{n-2}}\,\bar{\psi}(x)\psi(x)h(x) \end{array} \tag{14}$$

This introduces
FAINT but Non-Polynomial "evanescent" (fading-out) interactions $\propto 2\epsilon = 4-n$

$$\Phi = Me^{\phi/M},\ \langle\Phi\rangle \equiv M \quad \to \quad [\Phi(x)]^{\frac{4-n}{n-2}} = M^{\frac{\epsilon}{1-\epsilon}}\left(1+\frac{\epsilon}{1-\epsilon}\frac{\phi}{M}++\frac{1}{2}(\frac{\epsilon}{1-\epsilon})^2\frac{\phi^2}{M^2}+\cdots\right) \tag{15}$$

This scenario would give quantum scale invariant theory, which might realize the vanishing CC.

5 Quantum scale-invariant renormalization

Explicit calculations were performed by

- 1-loop: D.M. Ghilencea, Phys.Rev. D93(2016)105006.
- 2-loop: Ghilencea, Lalak and Olszewski, Eur.Phys.J. C(2016)76:656.
- 2.5-loop: Ghilencea, Phys.Rev. D97(2018)075015.
- c.f. RGE: C. Tamarit, JHEP 12(2013)098.

in a simple 2-scalar model: $(h \ \to \ \phi, \ \Phi \ \to \ \sigma)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi + \frac{1}{2}\partial_\mu\sigma \cdot \partial^\mu\sigma - V(\phi,\sigma) \tag{16}$$

with scale-invariant potential in n dimension:

$$V(h,\Phi) = \mu(\sigma)^{4-n}\left(\frac{\lambda_\phi}{4}\phi^4 - \frac{\lambda_m}{2}\phi^2\sigma^2 + \frac{\lambda_\sigma}{4}\sigma^4\right) \tag{17}$$

with

$$\mu(\sigma) = z\sigma^{\frac{2}{n-2}} \qquad (z : \text{renormalization point parameter}) \tag{18}$$

At tree level, $\lambda_m^2 = \lambda_\phi\lambda_\sigma$ is assumed so that

$$V(\phi,\sigma) = \mu(\sigma)^{4-n}\frac{\lambda_\phi}{4}\left(\phi^2 - \varepsilon\sigma^2\right)^2$$

$$\lambda_m = \varepsilon\lambda_\phi, \qquad \lambda_\sigma = \varepsilon^2\lambda_\phi \tag{19}$$

Ghilencea has shown:

1. Non-renormalizability: higher and higher order non-polynomial interaction terms of the form
$$\frac{\phi^{4+2p}}{\sigma^{2p}} \quad (p = 1, 2, 3, \cdots) \tag{20}$$
are induced by the evanescent interactions at higher loop level, and they must also be included as counterterms, can be neglected in the low-energy region $E < \langle\sigma\rangle \sim M_{\rm Pl}$.

2. Mass hierarchy is stable: If we put
$$\lambda_\phi = \bar{\lambda}_\phi, \quad \lambda_m = \varepsilon\bar{\lambda}_m, \quad \lambda_\sigma = \varepsilon^2\bar{\lambda}_\sigma \tag{21}$$
with $\bar{\lambda}_i$'s ($i = \phi, m, \sigma$): $O(1)$ and very tiny $\varepsilon = \left(\frac{100\text{GeV}}{10^{18}\text{GeV}}\right)^2 = 10^{-32}$, then, $\bar{\lambda}_i$'s remain $O(1)$ stably against radiative corrections. This is essentially because $\sigma^2\phi^2$ term comes only through the $\lambda_m\phi^2\sigma^2$ interaction.

One-loop potential at $n = 4$: scale Invariant!
$$\begin{aligned} V(\phi,\ \sigma) &= \frac{\lambda_\phi}{4}\phi^4 - \frac{\lambda_m}{2}\phi^2\sigma^2 + \frac{\lambda_\sigma}{4}\sigma^4 \\ &\quad + \frac{\hbar}{64\pi^2}\left\{M_1^4\left(\ln\frac{M_1^2}{z^2\sigma^2} - \frac{3}{2}\right) + M_2^4\left(\ln\frac{M_2^2}{z^2\sigma^2} - \frac{3}{2}\right) + \Delta V\right\} \end{aligned} \tag{22}$$
$$\begin{aligned} \Delta V &= -\lambda_\phi\lambda_m\frac{\phi^6}{\sigma^2} + (16\lambda_\phi\lambda_m - 6\lambda_m^2 + 3\lambda_\phi\lambda_\sigma)\phi^4 \\ &\quad + (-16\lambda_m + 25\lambda_\sigma)\lambda_m\phi^2\sigma^2 - 21\lambda_\sigma^2\sigma^4 \end{aligned} \tag{23}$$

However, the problem, (which Ghilencia has not mentioned to), is that

3. **Vanishing CC again requires fine tuning!** owing to quantum corrections.
$V(\phi,\sigma) = \sigma^4 W(x)$ with $x \equiv \phi^2/\sigma^2$.

Since the stationarity
$$\begin{cases} \phi\dfrac{\partial}{\partial\phi}V &= \sigma^4 W'(x)\cdot 2x &= 0 \\ \sigma\dfrac{\partial}{\partial\sigma}V &= \sigma^4\Big(4W(x) + W'(x)\cdot(-2x)\Big) &= 0 \end{cases} \tag{24}$$

requires
$$W'(x) = 0 \text{ and } W(x) = 0 \quad \text{are satisfied.} \tag{25}$$

Let us examine these conditions with the above 1-loop potential

$$\begin{aligned} W(x) &= \frac{\lambda_\phi}{4}x^2 - \frac{\lambda_m}{2}x + \frac{\lambda_\sigma}{4} \\ &\quad + \frac{\hbar}{64\pi^2}\left\{\frac{M_1^4}{\sigma^4}\left(\ln\frac{M_1^2}{z^2\sigma^2} - \frac{3}{2}\right) + \frac{M_2^4}{\sigma^4}\left(\ln\frac{M_2^2}{z^2\sigma^2} - \frac{3}{2}\right)\right. \\ &\quad\quad \left. - \lambda_\phi\lambda_m x^3 + (16\lambda_\phi\lambda_m - 6\lambda_m^2 + 3\lambda_\phi\lambda_\sigma)x^2 + (-16\lambda_m + 25\lambda_\sigma)\lambda_m x - 21\lambda_\sigma^2\right\} \end{aligned}$$

At tree level, the stationary point $x = x_0$ is

$$\begin{cases} W'(x_0) = \dfrac{\lambda_\phi}{2}x_0 - \dfrac{\lambda_m}{2} = 0 \;\rightarrow\; x_0 = \dfrac{\lambda_m}{\lambda_\phi} \\ W(x_0) = \dfrac{\lambda_\phi}{4}x_0^2 - \dfrac{\lambda_m}{2}x_0 + \dfrac{\lambda_\sigma}{4} = 0 \;\rightarrow\; \lambda_\sigma = \dfrac{\lambda_m^2}{\lambda_\phi} \end{cases} \tag{26}$$

Note that $W'(x) = 0$ determined $x = \frac{\langle\phi\rangle^2}{\langle\sigma\rangle^2}$, but $W(x) = 0$ imposed a constraint on λ_i's.

At one-loop level, the stationary point may be shifted and the coupling constants may be adjusted:

$$x = x_0 + \hbar x_1, \qquad \lambda_i \;\Rightarrow\; \lambda_i + \hbar\,\delta\lambda_i \;\; (i = \phi, m, \sigma) \tag{27}$$

$W'(x) = 0$ requires, at $O(\hbar)$,

$$\begin{aligned} W'(x)\Big|_{O(\hbar)} &= \frac{\lambda_\phi}{2}x_1 + \frac{\delta\lambda_\phi}{2}x_0 + \frac{\delta\lambda_m}{2} \\ &\quad + \frac{1}{64\pi^2}\left[4\lambda_\phi\lambda_m(3 + 2x_0 - x_0^2)\left(\ln\frac{2\lambda_m(1+x_0)}{z^2} - 1\right) + 16\lambda_m^2(1+x_0)\right] \end{aligned}$$

$\rightarrow$ consistent with the VEV (mass) hierarchy; i.e., no fine tuning necessary

$$x = \frac{\langle\phi\rangle^2}{\langle\sigma\rangle^2} = O(\varepsilon) \;\text{ since }\; \lambda_m, \delta\lambda_m \sim O(\varepsilon),\; \lambda_\phi, \delta\lambda_\phi \sim O(1), \rightarrow\; x_{0,1} \sim O(\varepsilon). \tag{28}$$

Next,

$$W(x)\Big|_{O(\hbar)} = \frac{\delta\lambda_\phi}{4}x_0^2 + \frac{\delta\lambda_m}{2}x_0 + \frac{\delta\lambda_\sigma}{4} + \frac{1}{64\pi^2}\left[4\lambda_m^2(1+x_0)^2\left(\ln\frac{2\lambda_m(1+x_0)}{z^2} - \frac{3}{2}\right)\right] \tag{29}$$

All the terms are consistently of $O(\varepsilon^2)$, so that $W(x) = 0$ is realized up to $o(\varepsilon^2)$ by $O(1)$ tuning of $\bar{\lambda}_\phi,\ \bar{\lambda}_m,\ \bar{\lambda}_\sigma$. However, the Vacuum Energy $\sigma^4 W(x)$ at the stationary point is made vanish only in the sense of $O(\varepsilon^2) \times \sigma^4 = O((100\text{GeV})^4)$.

If we require the vanishingness up to the order of $\Lambda_0 \sim (1\text{meV})^4 \sim 10^{-56} \times (100\text{GeV})^4$, then, we have still to tune $\bar{\lambda}_\phi,\ \bar{\lambda}_m,\ \bar{\lambda}_\sigma$ in 56 digits!

$$\boxed{\text{We still need Superfine Tuning even in quantum Scale-Invariant theory}} \tag{30}$$

This is nothing but the original CC problem!

Quantum SI is not enough to solve the CC problem.

Note also, however, that this is also the problem beyond the perturbation theory. We are discussing the Vacuum energy in much much finer precision than the purturbation expansion parameter $O(\hbar/16\pi^2)$.

6 What happens?

If the theory is quantum scale-invariant, then

$$\sum_i \phi_i \frac{\partial}{\partial \phi_i} V(\phi) = 4V(\phi) \tag{31}$$

implying $V(\phi_i^0) = 0$ at any stationary point ϕ_i^0, and any point in that direction, $\rho\phi_i^0$ with $\forall\rho \in \mathbf{R}$ also realizes the vanishing energy $V(\rho\phi_i^0) = \rho^4 V(\phi_i^0) = 0$. (flat direction)

If $V(\phi) \neq 0$ at $\exists\phi$, then the potential is not stationary at ϕ.

In the above: $V(\phi,\sigma) = \sigma^4 W(x)$ was flat in the direction x_0 at the tree level, $W(x_0)$=0, but, at one-loop, did not exactly satisfy $W(x_0 + \hbar x_1) = 0$ at the 'stationary point' realizing $W'(x_0 + \hbar x_1) = 0$ exactly, unless the coupling constants were superfine-tuned.

This means from the above Eq. (24) that the point $x_0 + \hbar x_1$ realizes the stationarity with respect to ϕ but not necessarily to σ. If $W(x_0 + \hbar x_1) = 0$ is not exactly satisfied, then the potential has a small gradient $\sigma(\partial/\partial\sigma)V = 4\sigma^4 W(x) = \sigma^4 O(\varepsilon^2) \neq 0$ in the σ-direction, implying that the potential is stationary only at the origin $\sigma = 0$!

$$\boxed{\text{The flat direction is lifted by the radiative correction}} \tag{32}$$

Quantum scale invariance alone does not protect the flat direction, automatically!

We need keep flat direction against quantum radiative corrections. We may still need other symmetry to realize flat directions. (SUSY?)

Recall; $\frac{\delta V}{\delta \phi} = 0$ and $\frac{\delta V}{\delta \sigma} = 0$ required, for $V = \sigma^4 W$, respectively,

$$(1) \quad W'(x;\lambda)\Big|_{x=x_0(\lambda)} = 0 \qquad \Rightarrow \qquad (2) \quad W\big(x_0(\lambda);\lambda\big) = 0$$

determines the VEV ratio $x = \dfrac{\phi^2}{\sigma^2} = x_0(\lambda)$ demands super fine tuning of λ's.

Tomboulis said interesting thing: He introduces ren. pt. μ in addition to the dilaton σ and consider the running of coupling constant:

$$\bar{\lambda}(z), \qquad z \equiv \frac{\mu}{\sigma} \ : \text{renormalization point parameter}$$

a la Ghilencea and C. Tamarit, JHEP12(2013)098 (33)

Then, he claims that, the second condition, now reading

$$W(x_0(\bar{\lambda}(z)); \bar{\lambda}(z)) = 0, \tag{34}$$

is now simply an equation determining the ren. pt. $z_0 = \mu_0/\sigma_0$ and so is automatically satisfied without any fine tuning.

This interesting idea, however, does not work unfortunately, since

$$\frac{d}{dz} W(x_0(\bar{\lambda}(z)); \bar{\lambda}(z)) = 0. \tag{35}$$

Changing the ren. pt. $z = \mu/\sigma$ cannot change the value of $W(x_0(\bar{\lambda}(z)); \bar{\lambda}(z))$.

7 Conclusion

I believe that the scale invariance is the right direction for solving CC problem, but it is still missing something.

Lesson: We need a symmetry or mechanism to realize

$$\begin{aligned} \text{Spontaneous SI breaking} &= \text{Non-vanishing field VEV} \\ &= \exists \text{ flat direction of } V \end{aligned}$$

Thank You

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1. **Running coupling explain the hierarchy** after VEV $\langle\sigma\rangle \neq 0$ appears.

   e.g., Chiral symmetry breaking scale in QCD:

   Usually the coupling $\alpha_3 \equiv g_3^2/4\pi$ runs according to

$$\mu\frac{d}{d\mu}\alpha_3(\mu) = 2b_3\,\alpha_3^2(\mu) \quad \to \quad \frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(M)} - b_3\ln\frac{\mu^2}{M^2}$$
$$\to \quad \frac{1}{\alpha_3^{\rm cr}} = \frac{1}{\alpha_3(M)} - b_3\ln\frac{\Lambda_{\rm QCD}^2}{M^2}$$

   where $\alpha_3^{\rm cr} = O(1)$ quantity like $\pi/3$, so explains the huge hierarchy:

$$\varepsilon = \frac{\Lambda_{\rm QCD}^2}{M^2} = \exp\frac{1}{b_3}\Big(\frac{1}{\alpha_3(M)} - \frac{1}{\alpha_3^{\rm cr}}\Big). \tag{36}$$

   This is the usual explanation.

   The following is still a handwaving argument to be confirmed.

   In quantum SI theory, $\alpha_3(M)$ here, probably, should be replaced by M-independent initial gauge coupling $\alpha_3^{\rm init}$, while the initial scale M^2 should be replaced by the dilaton field VEV $\langle\sigma\rangle^2$. Then

$$\frac{1}{\alpha_3^{\rm cr}} - \frac{1}{\alpha_3^{\rm init}} = -b_3\ln\frac{\Lambda_{\rm QCD}^2}{\langle\sigma\rangle^2} \tag{37}$$

   so that the QCD scale $\Lambda_{\rm QCD}$ is always scaled with the dilaton VEV $\langle\sigma\rangle$.

2. **Hierarchy and Effective Potential**

This hierarchy should show up in the effective potential. And the effective potential should be calculable prior to the spontaneous breaking.

Since Λ_{QCD}^2 should stand for the VEV $\varphi^\dagger\varphi$ of the chiral sigma model scalar field φ, we suspect that we should be able to derive the effective potential of the Coleman-Weinberg type like

$$\frac{(\varphi^\dagger\varphi)^2}{64\pi^2}\left(-b_3\ln\frac{\varphi^\dagger\varphi}{\sigma^2}+\frac{1}{\alpha_3^{\mathrm{init}}}-\frac{1}{\alpha_3^{\mathrm{cr}}}\right)^2 \tag{38}$$

8 Can dynamical breaking of SI occur?

We may still need other symmetry to realize flat directions. (SUSY?)

Or, we need dynamical breaking of scale-invariance in quantum scale-invariant theory. If the scale invariance can be broken dynamically, i.e., $\phi \neq 0$, then $V = 0$ is automatic in any case.

So, let us examine (quantum) SI NJ-L model:

$$\begin{aligned}\mathcal{L} &= \bar{\psi} i\gamma^\mu \partial_\mu \psi + \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{\lambda}{4!}\phi_0^4 + \frac{g^2}{N\phi_0^2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5 \psi)^2\right] \\ &\Rightarrow \bar{\psi} i\gamma^\mu \partial_\mu \psi + \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{\lambda}{4!}\phi_0^4 + \frac{N}{4}\phi_0^2(\sigma^2+\pi^2) - g\bar{\psi}(\sigma + i\gamma_5 \pi)\psi.\end{aligned}$$

supplemented with the UV cut-off set at $\Lambda^2 = \phi_0^2$. $\phi_0 = \text{dilaton} \times \varepsilon$.

The $1/N$-leading potential is found to be

$$\begin{aligned}V(\sigma, \pi, \phi_0) &= \frac{N}{4}\phi_0^2(\sigma^2+\pi^2) + \frac{\lambda}{4!}\phi_0^4 - 2N\int_0^{\phi_0^2} \frac{d^4 k_\mathrm{E}}{(2\pi)^4} \ln[k_\mathrm{E}^2 + g^2(\sigma^2+\pi^2)] \\ &= N\phi_0^2\left\{\frac{1}{2}\frac{x}{g^2} + \frac{\lambda}{4N} - \frac{1}{8\pi^2}\cdot \underbrace{\frac{1}{2}\left[\ln(1+x) - x^2\ln\left(1+\frac{1}{x}\right) + x\right]}_{\equiv f(x)}\right\}\end{aligned}$$

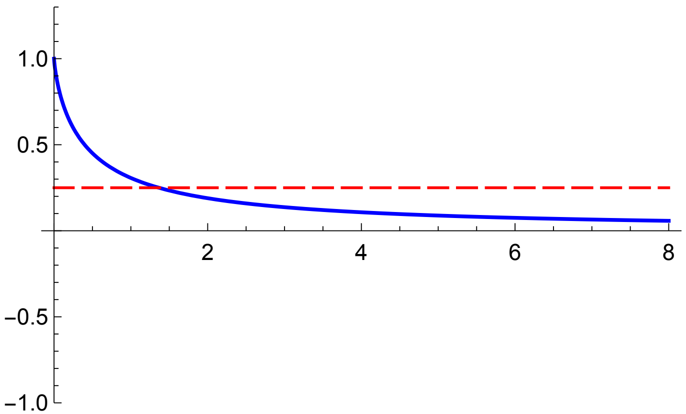
with $x \equiv g^2(\sigma^2+\pi^2)/\phi_0^2$.

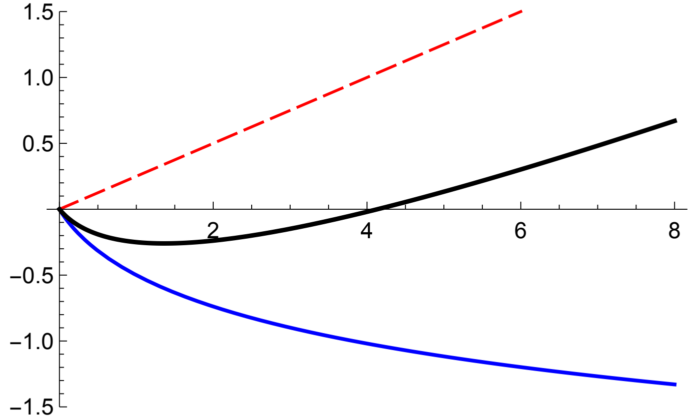
Note the structure $\boxed{V = N\phi_0^4\, v(x; g, \lambda).}$

Stationary conditions:

$$\text{a)} \quad \frac{\partial v}{\partial x} = 0 \;\Rightarrow\; \frac{4\pi^2}{g^2} = \underbrace{f'(\bar{x})}_{\text{figure}\downarrow} \qquad\qquad \text{b)} \quad \frac{\partial V}{\partial \phi_0} = 0 \;\Rightarrow\; \phi_0 = 0 \;\; \text{or}$$

$$8\pi^2 v(\bar{x}) = \underline{\frac{4\pi^2}{g^2}\bar{x} - f(\bar{x})} + \frac{2\pi^2}{N}\lambda = 0$$

a) 

b) 

So, still here, $v(\bar{x}) = 0$ is achieved only with fine tuning of coupling constants g, λ.

Driving force of causing spontaneous symmetry breaking (chiral and scale invariance) only works to make a minimum of $v(x)$ at $x \neq 0$, but not to make a flat direction i.e., direction of $v(x) = 0$.