Homomorphisms of pseudo-Riemannian calculi and noncommutative minimial submanifolds

Joakim Arnlind
Linköping University

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References

- **Noncommutative minimal embeddings and morphisms of pseudo-Riemannian calculi** (`arXiv:1906:03885`)
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- **Riemannian curvature of the noncommutative 3-sphere** (J. Noncommut. Geom. 2017)
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- **On the Chern-Gauss-Bonnet theorem for the noncommutative 4-sphere** (J. Geom. Phys. 2016)
  J.A. and M. Wilson

Introduction

- For a number of years, we've been interested in connections and curvature of noncommutative manifolds and, initially, we wanted to better understand the concept of a torsion-free and metric (Levi-Civita) connection in NCG.
- We start from data consisting of a $*$-algebra, a module ("vector fields") and a Lie algebra of derivations. Whatever approach to a derivation based calculus one takes, these object will probably appear.
- Given this data, we asked the question: What kind of assumptions give the uniqueness of a Levi-Civita connection?
- We collected these assumptions into the concept of "pseudo-Riemannian caluli".
- We considered several examples that fit into the framework (e.g. noncommutative torus, noncommutative spheres) and explicitly constructed the Levi-Civita connection and computed its curvature.

Introduction

- Moreover, for the noncommutative 4-sphere, we could prove a Chern-Gauss-Bonnet type theorem by constructing the Pfaffian of the curvature form and computing its integral.
- Moreover, we recently started to study these objects from a more algebraic perspective, starting by considering morphisms of real calculi.
- The concept of a morphism opened up for defining noncommutative embeddings, and we showed that there exists a nice theory of embeddings containing analogues of classical objects such as the second fundamental form, Weingarten's map and Gauss' equations.
- We propose a definition of mean curvature and, consequently, of minimal embeddings. As an example of the new concepts we show that the noncommutative torus can be minimally embedded into the noncommutative 3-sphere.

A few references

This work is in the tradition of derivation based differential calculus on noncommutative algebras.

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Pseudo-Riemannian calculi

Let us now recall the concept of a real calculus as well as pseudo-Riemannian calculi.

The idea is to naively copy the basic algebraic structures of Riemannian geometry to the noncommutative case:

- $\mathcal{A}$ – noncommutative $*$-algebra (complex valued functions)
- M – projective (right) $\mathcal{A}$-module (vector fields)
- h – $\mathcal{A}$-bilinear map $h : M \times M \to \mathcal{A}$ (metric)
- $\nabla : \mathrm{Der}(\mathcal{A}) \times M \to M$ (connection)
- $\varphi : \mathrm{Der}(\mathcal{A}) \to M$

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- $\nabla : \mathrm{Der}(\mathcal{A}) \times M \to M$ (connection)
- $\varphi : \mathrm{Der}(\mathcal{A}) \to M$

In differential geometry, if one chooses M to be the vector fields, φ is the isomorphism between derivations and vector fields; in this context we only require that each derivation corresponds to a vector field (but not necessarily the other way around). Let us now make these concepts more precise.

Real (metric) calculus

Definition

Let $\mathcal{A}$ be a $*$-algebra, M be a (right) $\mathcal{A}$-module, $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a (real) Lie algebra of hermitian derivations and let $\varphi : \mathfrak{g} \to M$ be a $\mathbb{R}$-linear map. The data $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is called a *real calculus* if the image $M_{\varphi} = \varphi(\mathfrak{g})$ generates M as a (right) $\mathcal{A}$-module,

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Definition

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is a real calculus and let h be a nondegenerate hermitian form on M. If

$$h(E_1, E_2)^* = h(E_1, E_2)$$

for all $E_1, E_2 \in \mathrm{Im}(\varphi)$, then $(C_{\mathcal{A}}, h)$ is called a *real metric calculus*.

We think of elements in $\mathrm{Im}(\varphi)$ as "real" vector fields.

Real connection calculus

Let us now add a connection to the previous data.

Definition

Let $(C_{\mathcal{A}}, h)$ be a real metric calculus and let $\nabla : \mathfrak{g} \times M \to M$ denote an affine connection on M. If it holds that

$$h(\nabla_d E_1, E_2) = h(\nabla_d E_1, E_2)^*$$

for all $E_1, E_2 \in M_\varphi$ and $d \in \mathfrak{g}$ then $(C_{\mathcal{A}}, h, \nabla)$ is called a *real connection calculus*.

We think of this condition as a reality condition on ∇.

Pseudo-Riemannian calculus

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Definition

Let $(C_{\mathcal{A}}, h, \nabla)$ be a real connection calculus over M. The calculus is *metric* if

$$d\big(h(U,V)\big) = h\big(\nabla_d U, V\big) + h\big(U, \nabla_d V\big)$$

for all $d \in \mathfrak{g}$, $U, V \in M$, and *torsionfree* if

$$\nabla_{d_1}\varphi(d_2) - \nabla_{d_2}\varphi(d_1) - \varphi\big([d_1, d_2]\big) = 0$$

for all $d_1, d_2 \in \mathfrak{g}$. A metric and torsionfree real connection calculus over M is called a *pseudo-Riemannian calculus over M*.

Uniqueness of the pseudo-Riemannian calculus

Given a real metric calculus, there is no guarantee that one may find a torsionfree and metric connection. The metric is assumed to be non-degenerate, but not in general invertible.

However, if such a connection exists, it is unique:

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However, if such a connection exists, it is unique:

Theorem

Let $(C_{\mathcal{A}}, h)$ be a real metric calculus over M. Then there exists at most one connection ∇ on M, such that $(C_{\mathcal{A}}, h, \nabla)$ is a pseudo-Riemannian calculus (i.e., such that ∇ is a real, torsionfree and metric connection).

(This result is obtained by deriving a Koszul formula for the connection.)

The noncommutative torus

The noncommutative torus T^2_θ is defined via two unitary generators U, V satisfying $VU = e^{i\theta}UV$. Introduce

$$X^1 = \frac{1}{2\sqrt{2}}(U^* + U) \qquad X^2 = \frac{i}{2\sqrt{2}}(U^* - U)$$

$$X^3 = \frac{1}{2\sqrt{2}}(V^* + V) \qquad X^4 = \frac{i}{2\sqrt{2}}(V^* - V)$$

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Let $\mathfrak{g}$ be the Lie algebra generated by the two canonical derivations δ_1, δ_2 on T^2_θ. M is the submodule of $(T^2_\theta)^4$ generated by

$$E_1 = \partial_1(X^1, X^2, X^3, X^4) = (-X^2, X^1, 0, 0)$$
$$E_2 = \partial_2(X^1, X^2, X^3, X^4) = (0, 0, -X^4, X^3)$$

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Define $\varphi : \mathfrak{g} \to M$ by $\varphi(\delta_i) = E_i$ for $i = 1, 2$. This defines a real calculus over the noncommutative torus. Furthermore, one can prove that M is a free module of rank 2.

The noncommutative 3-sphere

We consider the 3-sphere as defined by K. Matsumoto: Let S^3_θ be the $*$-algebra generated by two normal elements Z, W satisfying

$$WZ = qZW \qquad W^*Z = \bar{q}ZW^* \qquad WW^* + ZZ^* = \mathbb{1},$$

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and introduce

$$X^1 = \frac{1}{2}(Z + Z^*) \qquad X^2 = \frac{1}{2i}(Z - Z^*)$$
$$X^3 = \frac{1}{2}(W + W^*) \qquad X^4 = \frac{1}{2i}(W - W^*),$$

implying $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \mathbb{1}$. Normality of Z, W is equivalent to $[X^1, X^2] = [X^3, X^4] = 0$.

The noncommutative 3-sphere

Let $\mathfrak{g}$ be the Lie algebra generated by the derivations

$$\begin{aligned} \partial_1(Z) &= iZ & \partial_1(W) &= 0 \\ \partial_2(Z) &= 0 & \partial_2(W) &= iW \\ \partial_3(Z) &= Z|W|^2 & \partial_3(W) &= -W|Z|^2, \end{aligned}$$

giving $[\partial_a, \partial_b] = 0$ for $a, b = 1, 2, 3$.

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giving $[\partial_a, \partial_b] = 0$ for $a, b = 1, 2, 3$.

Let M be the submodule of $(S^3_\theta)^4$ generated by

$$\begin{aligned} E_1 &= (-X^2, X^1, 0, 0) \\ E_2 &= (0, 0, -X^4, X^3) \\ E_3 &= (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2), \end{aligned}$$

where $|Z|^2 = ZZ^*$ and $|W|^2 = WW^*$. One easily proves that M is a free module with basis E_1, E_2, E_3. Furthermore, set $\varphi(\partial_a) = E_a$.

The noncommutative 3-sphere

Define

$$h(U, V) = (U^a)^* h_{ab} V^b$$

where

$$h_{ab} = \sum_{k=1}^{4} (E_a^k)^* E_b^k = \begin{pmatrix} |Z|^2 & 0 & 0 \\ 0 & |W|^2 & 0 \\ 0 & 0 & |Z|^2|W|^2 \end{pmatrix}.$$

The above data defines a real metric calculus, and one may compute the (unique) Levi-Civita connection as

$$\begin{aligned} &\nabla_{\partial_1} E_1 = -E_3 \quad && \nabla_{\partial_2} E_2 = E_3 \quad && \nabla_{\partial_3} E_3 = E_3(|W|^2 - |Z|^2) \\ &\nabla_{\partial_1} E_2 = 0 && \nabla_{\partial_1} E_3 = E_1|W|^2 && \nabla_{\partial_2} E_3 = -E_2|Z|^2. \end{aligned}$$

Curvature of the 3-sphere

One may proceed to compute the curvature operators

$$R(\partial_a,\partial_b)U = \nabla_{\partial_a}\nabla_{\partial_b}U - \nabla_{\partial_b}\nabla_{\partial_a}U - \nabla_{[\partial_a,\partial_b]}U$$

$$R(\partial_1,\partial_2) = \begin{pmatrix} 0 & |W|^2 & 0 \\ -|Z|^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(\partial_1,\partial_3) = \begin{pmatrix} 0 & 0 & |Z|^2|W|^2 \\ 0 & 0 & 0 \\ -|Z|^2 & 0 & 0 \end{pmatrix}$$

$$R(\partial_2,\partial_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & |Z|^2|W|^2 \\ 0 & -|W|^2 & 0 \end{pmatrix}$$

Morphisms of real calculi

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Definition

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ and $C_{\mathcal{A}'} = (\mathcal{A}', \mathfrak{g}', M', \varphi')$ be real calculi and assume that $\phi : \mathcal{A} \to \mathcal{A}'$ is a $*$-algebra homomorphism. If there is a Lie algebra homomorphism $\psi : \mathfrak{g}' \to \mathfrak{g}$ such that

$$\delta(\phi(a)) = \phi(\psi(\delta)(a)) \text{ for all } \delta \in \mathfrak{g}', a \in \mathcal{A}$$

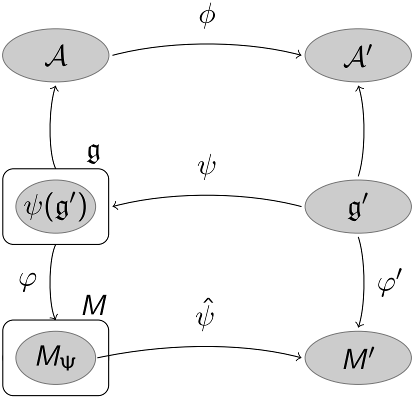
and a map $\hat{\psi} : M_\Psi \to M'$ such that

1. $\hat{\psi}(m_1 + m_2) = \hat{\psi}(m_1) + \hat{\psi}(m_2)$ for all $m_1, m_2 \in M$
2. $\hat{\psi}(ma) = \hat{\psi}(m)\phi(a)$ for all $m \in M$ and $a \in \mathcal{A}$
3. $\hat{\psi}(\Psi(\delta)) = \varphi'(\delta)$ for all $\delta \in \mathfrak{g}'$,

then $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is called a morphism of real calculi, where $\Psi = \varphi \circ \psi$ and $M_\Psi \subseteq M$ is the image of Ψ.

Let us illustrate the above definition with a picture.

"Commuting" diagram of a morphism of real calculi



$$\Psi = \varphi \circ \psi : \mathfrak{g}' \to M$$

Compare the above diagram with a manifold Σ' embedded in Σ.

ψ – Extension of vector fields on Σ' to vector fields on Σ

$\hat{\psi}$ – Restriction of vector fields on Σ tangent to Σ'

Morphims of real metric calucli

Definition

Let $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ be real metric calculi and assume that $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is a real calculus homomorphism. If

$$h'\big(\varphi'(\delta_1), \varphi'(\delta_2)\big) = \phi\big(h(\Psi(\delta_1), \Psi(\delta_2))\big)$$

for all $\delta_1, \delta_2 \in \mathfrak{g}'$ then $(\phi, \psi, \hat{\psi})$ is called a *real metric calculus homomorphism*.

Embeddings

Definition

A homomorphism of real calculi $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is called an *embedding* if ϕ is surjective and there exists a submodule $\tilde{M} \subseteq M$ such that $M = M_\Psi \oplus \tilde{M}$. A homomorphism of real metric calculi $(\phi, \psi, \hat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$ is called an *isometric embedding* if $(\phi, \psi, \hat{\psi})$ is an embedding and $M = M_\Psi \oplus M_\Psi^\perp$.

In analogy with classical Riemannian submanifold theory, one decomposes the Levi-Civita connection of the embedded manifold in its tangential and normal parts.

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$$\nabla_{\psi(\delta)} m = L(\delta, m) + \alpha(\delta, m) \tag{1}$$

$$\nabla_{\psi(\delta)} \xi = -A_{\xi}(\delta) + D_{\delta}\xi \tag{2}$$

for $\delta \in \mathfrak{g}'$, $m \in M_{\Psi}$ and $\xi \in M_{\Psi}^{\perp}$, with

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for $\delta \in \mathfrak{g}'$, $m \in M_{\Psi}$ and $\xi \in M_{\Psi}^{\perp}$, with

$$L(\delta, m) = P(\nabla_{\psi(\delta)} m) \qquad \alpha(\delta, m) = \Pi(\nabla_{\psi(\delta)} m)$$

$$A_{\xi}(\delta) = -P(\nabla_{\psi(\delta)} \xi) \qquad D_{\delta}\xi = \Pi(\nabla_{\psi(\delta)} \xi),$$

where $P : M \to M$ denotes the projection of $M = M_{\Psi} \oplus M_{\Psi}^{\perp}$ onto M_{Ψ}. The map $\alpha : \mathfrak{g}' \times M_{\Psi} \to M_{\Psi}^{\perp}$ is called the *second fundamental form* and $A : \mathfrak{g}' \times M_{\Psi}^{\perp} \to M_{\Psi}$ the *Weingarten map*.

Proposition

$L(\delta, m) = P(\nabla_{\psi(\delta)} m)$ is the Levi-Civita connection of the embedded manifold. (Or, more precisely, an extension of it to the ambient manifold.)

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Proposition

If $\delta_1, \delta_2 \in \mathfrak{g}'$, $a_1, a_2 \in \mathcal{A}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ then

$$\alpha\big(\delta_1, \Psi(\delta_2)\big) = \alpha\big(\delta_2, \Psi(\delta_1)\big)$$
$$\alpha\big(\lambda_1\delta_1 + \lambda_2\delta_2, m_1\big) = \lambda_1\alpha(\delta_1, m_1) + \lambda_2\alpha(\delta_2, m_1)$$
$$\alpha(\delta_1, m_1 a_1 + m_2 a_2) = \alpha(\delta_1, m_1)a_1 + \alpha(\delta_1, m_2)a_2$$

for $m_1, m_2 \in M_\Psi$.

Proposition

If $\delta \in \mathfrak{g}'$, $m \in M_\Psi$ and $\xi \in M_\Psi^\perp$ then $h\big(A_\xi(\delta), m\big) = h\big(\xi, \alpha(\delta, m)\big)$

Gauss' equation

Gauss' equation relates the curvature of the embedded manifold to the curvature of the ambient manifold via the second fundamental form.

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Proposition

Let $\delta_i \in \mathfrak{g}'$, $\partial_i = \psi(\delta_i) \in \mathfrak{g}$, $E_i = \Psi(\delta_i) \in M_\Psi$ *and* $E'_i = \varphi'(\delta_i) \in M'$ *for* $i = 1, 2, 3, 4$. *Then*

$$\begin{aligned} \phi\big(h(E_1, R(\partial_3,\partial_4)E_2)\big) &= h'\big(E'_1, R'(\delta_3,\delta_4)E'_2\big) \\ &\quad + \phi\left(h(\alpha(\delta_4,E_1),\alpha(\delta_3,E_2))\right) - \phi\left(h(\alpha(\delta_3,E_1),\alpha(\delta_4,E_2))\right). \end{aligned} \tag{3}$$

Minimal embeddings

- Recall that a minimal embedding (i.e. an embedding such that the induced metric minimizes the area of the embedded manifold) can be characterized by *zero mean curvature*. The mean curvature, is in it simplest form (codimension 1) the trace of the second fundamental form.

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Minimal embeddings

- Recall that a minimal embedding (i.e. an embedding such that the induced metric minimizes the area of the embedded manifold) can be characterized by *zero mean curvature*. The mean curvature, is in it simplest form (codimension 1) the trace of the second fundamental form.
- Having the second fundamental form at hand in noncommutative geometry suggests a natural definition of a noncommutative minimal embedding.
- Let us present a general construction as well as an example where the noncommutative torus is minimally embedded in the noncommutative 3-sphere.

Free real metric calculi

Definition

A real calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is called *free* if there exists a basis $\partial_1, ..., \partial_m$ of $\mathfrak{g}$ such that $\varphi(\partial_1), ..., \varphi(\partial_m)$ is a basis of M as a (right) $\mathcal{A}$-module.

Definition

A real metric calculus $(C_{\mathcal{A}}, h)$ is called *free* if $C_{\mathcal{A}}$ is free and h is invertible.

Invertible implies that $h_{ij} = h(\varphi(\partial_i), \varphi(\partial_j))$ is invertible as a matrix whenever $\partial_1, \ldots, \partial_m$ is a basis of $\mathfrak{g}$.

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Proposition

Let $(C_{\mathcal{A}}, h)$ be a free real metric calculus. Then there exists a unique affine connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is a pseudo-Riemannian calculus.

Mean curvature and minimal embeddings

Definition

Let $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ be free real metric calculi and let $(\phi, \psi, \hat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$ be an isometric embedding. Given a basis $\{\delta_i\}_{i=1}^{m'}$ of $\mathfrak{g}'$, the *mean curvature* $H_{\mathcal{A}'} : M \to \mathcal{A}'$ of the embedding is defined as

$$H_{\mathcal{A}'}(m) = \phi\left(h\big(m, \alpha(\delta_i, \Psi(\delta_j))\big)\right) h'^{ij}, \tag{4}$$

giving trivially $H_{\mathcal{A}'}(m) = 0$ for $m \in M_\Psi$. An embedding is called *minimal* if $H_{\mathcal{A}'}(\xi) = 0$ for all $\xi \in M_\Psi^\perp$.

(One easily prove that the above definition is independent of the basis chosen.)

Minimal embedding of the torus in S^3

As an example of the concepts introduced, let us construct a minimal embedding of the noncommutative torus in the noncommutative 3-sphere, in analogy with the classical case. In this context we shall consider a slightly more general metric on the 3-sphere:

$$h_{ab} = H \begin{pmatrix} |Z|^2 & 0 & 0 \\ 0 & |W|^2 & 0 \\ 0 & 0 & |Z|^2|W|^2 \end{pmatrix} H^*.$$

with $H \in S^3_\theta$ such that HH^* is invertible.

Furthermore, we localize the algebra of the 3-sphere to include the inverses of $|Z|^2$ and $|W|^2$.

Minimal embedding of the torus in S^3

Let us now construct the embedding $(\phi, \psi, \hat{\psi})$ of the noncommutative torus into the noncommutative 3-sphere. Set

$$\phi(Z) = \lambda U \quad \text{and} \quad \phi(W) = \mu V,$$

where λ and μ are complex nonzero constants such that $|\lambda|^2 + |\mu|^2 = 1$. It is easy to verify that with these conditions ϕ is a $*$-algebra homomorphism. Moreover, since λ and μ are chosen to be nonzero it means that ϕ is surjective as well.

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$$\psi(\delta_1) = \partial_1 \quad \text{and} \quad \psi(\delta_2) = \partial_2,$$

Furthermore, with

$$\hat{\psi}(E_1) = e_1 \quad \text{and} \quad \hat{\psi}(E_2) = e_2$$

$(\phi, \psi, \hat{\psi})$ is a real calculus homomorphism.

Minimal embedding of the torus in S^3

Recall that $(\phi, \psi, \hat{\psi})$ is an *embedding* if M (with basis E_1, E_2, E_3) splits into a direct sum $M = M_\Psi \oplus \tilde{M}$, where M_Ψ is the image of $\varphi \circ \psi$. In this case M_Ψ is the module generated by E_1, E_2 and $\tilde{M}$ is the module generated by E_3. Morphism Note that for any diagonal metric on M, the decomposition above is orthogonal.

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Next, we proceed to compute second fundamental form and the mean curvature of the embedding.

The mean curvature

The second fundamental form of the embedding is be computed as

$$\begin{aligned}
\alpha(\delta_1, \Psi(\delta_1)) &= -E_3(|W|^{-2}H_3 + \mathbb{1}) \\
\alpha(\delta_1, \Psi(\delta_2)) &= \alpha(\delta_2, \Psi(\delta_1)) = 0 \\
\alpha(\delta_2, \Psi(\delta_2)) &= E_3(\mathbb{1} - |Z|^{-2}H_3),
\end{aligned}$$

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with $H_a = \frac{1}{2}(HH^*)^{-1}\partial_a(HH^*)$ for $a = 1, 2, 3$, giving

$$\begin{aligned}H_{T^2_\theta}(m) &= \phi\left(h\big(m, \alpha(\delta_1, \Psi(\delta_1))\big)\right)(h')^{11} + \phi\left(h\big(m, \alpha(\delta_2, \Psi(\delta_2))\big)\right)(h')^{22}\\ &= \phi\left(h\big(m, -E_3(|W|^{-2}H_3 + \mathbb{1})\big)\right)|\lambda|^{-2}(\tilde{H}\tilde{H}^*)^{-1}\\ &\quad + \phi\left(h\big(m, E_3(\mathbb{1} - |Z|^{-2}H_3)\big)\right)|\mu|^{-2}(\tilde{H}\tilde{H}^*)^{-1}\\ &= \phi\left(h\big(m, E_3\big)\right)\left(|\mu|^{-2} - |\lambda|^{-2} - 2|\lambda|^{-2}|\mu|^{-2}\tilde{H}_3\right)(\tilde{H}\tilde{H}^*)^{-1},\end{aligned}$$

where $\tilde{H} = \phi(H)$.

Minimal embedding

The mean curvature:

$$H_{T^2_\theta}(m) = \phi\left(h(m, E_3)\right)\left(|\mu|^{-2} - |\lambda|^{-2} - 2|\lambda|^{-2}|\mu|^{-2}\tilde{H}_3\right)(\tilde{H}\tilde{H}^*)^{-1}$$

Hence, the (noncommutative) embedding of the torus into the 3-sphere is minimal if and only if

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In the special case where $\phi(\partial_3(HH^*)) = 0$, the embedding is minimal if $|\lambda| = |\mu| = 1/\sqrt{2}$ (in analogy with the classical case).

Summary

- One can develop a noncommutative submanifold theory much in analogy with classical differential geometry, giving the Weingarten's map, the second fundamental form as well as Gauss' equation (relating the curvature of the ambient manifold to the curvature of the embedded manifold).

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- With the help of the second fundamental form, one can define the mean curvature and, consequently, a noncommutative minimal embedding.
- As an example of these new concepts, we constructed a noncommutative minimal embedding of the torus into the 3-sphere.
- We hope that our (naive) considerations shed light on Riemannian submanifolds in noncommutative geometry, and what kind of results to expect.

Thanks for listening!