Hasse diagrams and Higgs branches

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Introduction: Hasse diagrams for symplectic singularities

In this note, we investigate the structure of Higgs branches of supersymmetric field theories with 8 supercharges. Standard arguments show that the Higgs branch X of such a theory is hyperKähler, and in the absence of mass terms and Fayet-Iliopoulos terms, it is singular. Singular hyperKähler varieties are called symplectic singularities. They admit a partition into a finite number of so-called symplectic leaves

$$X = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n \tag{1}$$

and these symplectic leaves are partially ordered using

$$\mathcal{O}_i \leq \mathcal{O}_j \qquad \Leftrightarrow \qquad \overline{\mathcal{O}_i} \subset \overline{\mathcal{O}_j}\,. \tag{2}$$

As for any partial order, this ordering of the leaves can be represented in a graphical form using a so-called Hasse diagram, which is our main object of interest [1].

The remainder of this note is organized as follows. In the first section, we explain what is the physical interpretation of the Hasse diagram, showing that it can be seen as partial Higgsing. In the second section, we demonstrate what methods can be used to compute these diagrams.

1 Higgs and Hasse

For brevity and concreteness, we focus in this section on a particular example, namely the four-dimensional $\mathcal{N} = 2$ theory with gauge group SU(4) and matter content made of one second rank antisymmetric tensor and 12 fundamental hypermultiplets. The Hasse diagram for this theory is displayed in Figure 1. We now explain how this diagram is interpreted, and what is the physics underlying it.

Each black dot represents one leaf. Here we have five distinct leaves. Next to it, the integer number is the quaternionic dimension of the leaf – and throughout this note, all dimensions will always be quaternionic. The top leaf, of dimension 39, is dense in the Higgs branch, and consists (by definition) of the generic points on the Higgs branch. Physically, they are field configurations which break entirely the SU(4) gauge group. Going down along the diagram, we learn that a subset of dimension 22 of the Higgs branch is singular inside the closure of the top leaf. Physically, this corresponds to non-complete Higgsing: on that subspace, SU(4) is broken to SU(2). It turns out that each leaf corresponds to one subgroup of SU(4), which is the subgroup left invariant by the hypermultiplet field configuration. This goes on until we reach the origin of the Higgs branch, which is the bottom leaf where the whole gauge group is unbroken. In the diagram, we indicate the unbroken gauge group next to the quaternionic dimension. Note that not all subgroups of SU(4) appear (for instance U(1) does not), this simply means that the matter content can only Higgs the gauge group to certain subgroups.

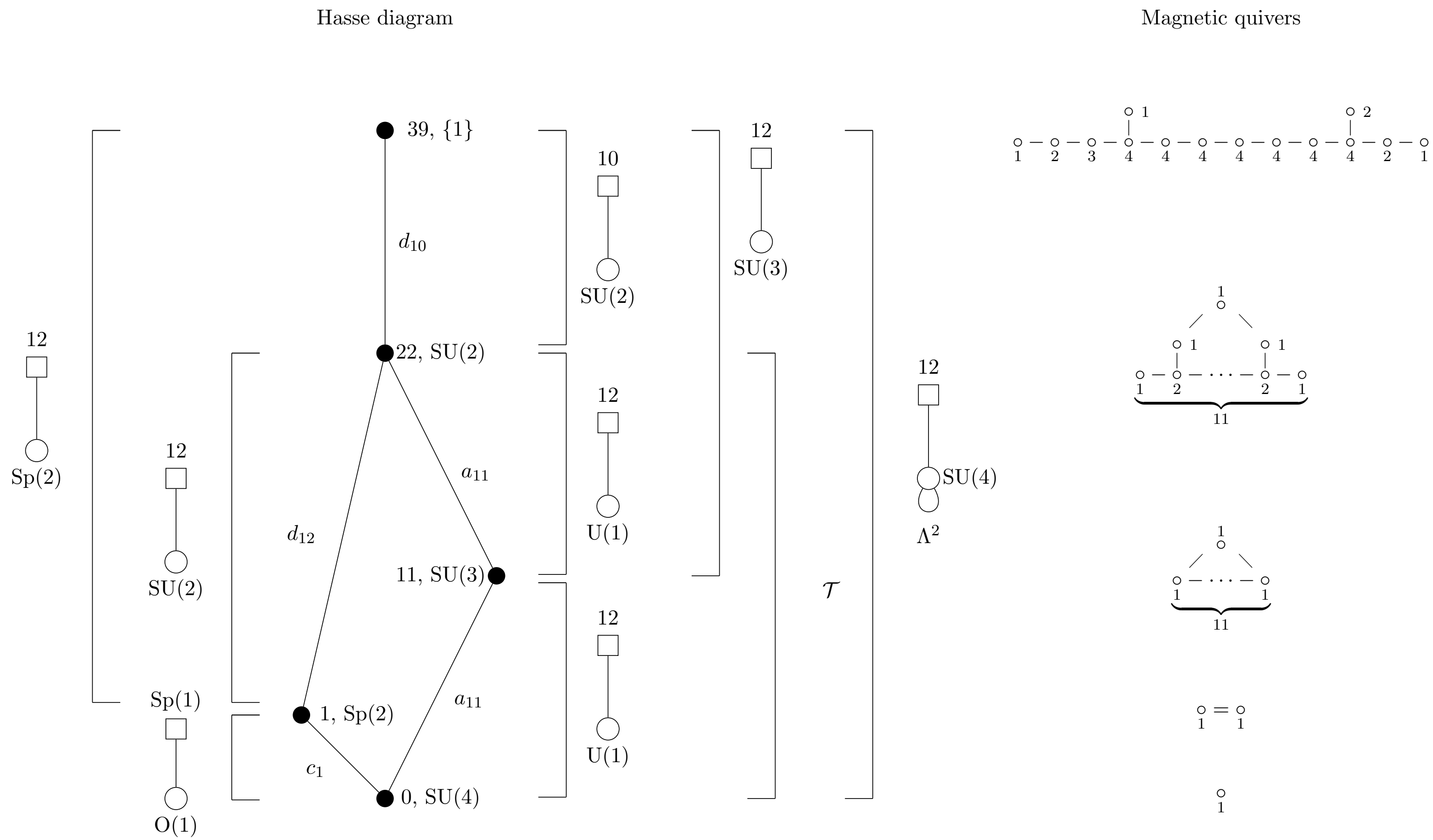

Figure 1: Hasse diagram for the SU(4) gauge theory with one second rank antisymmetric matter (denoted by Λ) and 12 fundamentals. For each leaf, denoted with a black dot, we give its quaternionic dimension next to it, and the associated magnetic quiver on the right. We also represent (using large brackets) electric quivers associated to every subdiagram between two points in the Hasse diagram. The $\mathcal{T}$ theory is a $U(2)$ gauge theory with twelve fundamentals of SU(2) with U(1) charge 1 and two SU(2) singlets of U(1) charge 2.

The Hasse diagram contains more information, located on the edges of the diagrams. Between to adjacent leaves in the partial order one can define a transverse slice. In many situations (though not all), these transverse slices belong to two families, first identified by Kraft and Procesi:

- Kleinian ADE singularities (of dimension 1), denoted A_n, D_n and E_n
- Closures of minimal nilpotent orbits of simple Lie algebras, denoted similarly but using lowercase (their dimension is $h^\vee - 1$).

In our example, we learn from the diagram that the transverse slice to the top leaf to the next is the 17-dimensional closure of the minimal orbit of $\mathfrak{so}(20, \mathbb{C})$. Note that the SU(2) theory with 10 flavors has exactly this as a Higgs branch. We represent it on the diagram using quiver notation, next to the d_{10} slice. Similarly, to any pair of comparable leaves one can associate a gauge theory which characterises the transverse slice. The Hasse diagrams for these theories are subdiagrams of the main Hasse diagram.

In each case, the non-abelian part of the global symmetry of a theory can be read from the lowest elementary slices. For our SU(4) for instance, the lowest elementary slices are c_1 and a_{11} giving a global symmetry including $\mathfrak{sp}(1) \oplus \mathfrak{su}(12)$.

2 Computing Hasse diagrams

Now that we have explained what physical information can be extracted from a Hasse diagram, we turn to how it is computed. The most elementary method is the standard partial Higgsing. To find the lowest non-trivial leaves, one enumerates all the maximal subgroups of the gauge group, and check which ones can be left as residual after giving a VEV to some hypermultiplets.

The above methods only works for Lagrangian theories. More generally, one can in many cases see a symplectic singularity as the 3d Coulomb branch of a quiver with unitary gauge nodes and bifundamental matter. Such a quiver, called a *magnetic quiver*, can be seen as a combinatorial way to encode the geometry of the singularity. The algorithm of *quiver subtraction* can then be used to derive mechanically the Hasse diagram, including the geometry of every elementary slice. In Figure 1 these diagrams are represented, for each leaf, on the right. Finally, when a brane realisation of the theory is available, it is possible to visualise geometrically the transitions from one leaf to another. This is the physical process underlying the quiver subtraction algorithm.

Using magnetic quivers and brane setups, one can go beyond perturbative Lagrangian theories. For instance, our SU(4) example has precisely the correct matter content to define, at strong coupling, a 6d SCFT. From the brane realisation, one can read that the Hasse diagram for its Higgs branch is identical to the one of Figure 1 with an additional e_8 (29 dimensional)slice on top. This is the generic effect of the small E_8 transition on a Higgs branch represented through its Hasse diagram : just add an e_8 transition on top.

3 The scope of Hasse diagrams

As we just saw, one of the virtues of representing Higgs branches through their Hasse diagrams is that it opens a window on non-perturbative regions of the moduli space, and give insights about the geometry there, in a simple and precise way. In the case of the small E_8 transition, we saw that although the physics is non-trivial, the modification of the Hasse diagram is straightforward. Other situations are richer: for instance, it turns out certain generalisations of Argyres-Douglas theories have Higgs branches associated to magnetic quivers forming complete graphs. This generically gives rise to very intricate Hasse diagrams, unveiling a delicate structure of partial Higgsings.

The global shape of the diagram is also relevant. For some theories, the Higgs branch is a union of several cones, and in that case the Hasse diagram has several maximal leaves (a leaf is maximal if it is contained in the closure of no other leaf). This is the case for SU(N) SQCD in the appropriate range of parameters.

Hasse diagram constitute a new tool to study Higgs branches, and more generally symplectic singularities. Several features remain to be explored. From the mathematical perspective, although the Kraft-Procesi elementary slices seem to appear prominently, they by no means exhaust the possible elementary transitions in a Hasse diagrams, and a complete classification appears to be out of reach at the moment. Leaves can also admit multiplicities, a point that deserves further study. From the physical perspective, it would be interesting to extend our tools to theories that don't admit unitary magnetic quivers, but only ortho-symplectic magnetic quivers (equivalently, brane configurations with orientifold planes). This would expand the scope of (non-Lagrangian) theories attainable by Hasse diagrams techniques, allowing for instance to understand the relations between fixed rank 4d $\mathcal{N} = 2$ SCFTs.

References

[1] A. Bourget, S. Cabrera, J. F. Grimminger, A. Hanany, M. Sperling, A. Zajac, and Z. Zhong. The Higgs Mechanism - Hasse Diagrams for Symplectic Singularities. 2019.