A graded geometric approach to E/M duality

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Motivation

Mathematical: Develop a geometric framework suitable for studying mixed-symmetry tensor fields $\Rightarrow$ Graded Geometry

Physical: Embed the E/M duality into this geometric setting.

- Provide a common starting point for the dualization of **p-form fields** and **mixed-symmetry tensors**.
- Unify different types of Abelian E/M duality, e.g. **standard** and **exotic** duality.

$\Rightarrow$ **Goal:** Find a universal geometric first-order parent Lagrangian!

Bipartite tensors in graded geometry

- Bipartite tensors generalize differential forms; their components contain two sets of antisymmetrized indices when expanded in some local coordinate system.
- We will consider the graded supermanifold $\mathcal{M} = T[1]M \oplus T[1]M$, where M is the D-dimensional Minkowski spacetime. $\mathcal{M}$ is eqquiped with the degree-0 coordinates x^i of M and two sets of degree-1 coordinates θ^i, χ^i satisfying:

$$\theta^i\theta^j = -\theta^j\theta^i\,, \qquad \chi^i\chi^j = -\chi^j\chi^i\,, \qquad \theta^i\chi^j = \chi^j\theta^i\,.$$

- There is an isomorphism between functions on $\mathcal{M}$ and bipartite tensor fields on M, namely $C^\infty(\mathcal{M})|_{p,q} \simeq \Omega^{p,q}(M)$. Thus, a (p,q) bipartite tensor living in $\Omega^{p,q}(M)$ can be expanded as

$$\omega_{p,q}(x,\theta,\chi) = \frac{1}{p!q!}\,\omega_{i_1\dots i_p|j_1\dots j_q}(x)\,\theta^{i_1}\dots\theta^{i_p}\chi^{j_1}\dots\chi^{j_q}$$

- If $\omega_{p,q} \in \Omega^{p,q}$ and $\zeta_{p',q'} \in \Omega^{p',q'}$ then

$$(\omega\zeta)_{p+p',q+q'} := \frac{1}{p!q!p'!q'!}\omega_{i_1\dots i_p|j_1\dots j_q}\zeta_{i_{p+1}\dots i_{p+p'}|j_{q+1}\dots j_{q+q'}}\,\theta^{i_1}\dots\theta^{i_{p+p'}}\chi^{j_1}\dots\chi^{j_{q+q'}}$$

and $(\omega\zeta)_{p+p',q+q'} \in \Omega^{p+p',q+q'}$.

Some useful maps - [de Medeiros, Hull '03]

- Exterior derivatives: $d = \theta^i \partial_i : \Omega^{p,q} \to \Omega^{p+1,q}$, $\widetilde{d} = \chi^i \partial_i : \Omega^{p,q} \to \Omega^{p,q+1}$
- Minkowski metric: $\eta = \eta_{ij}\, \theta^i \chi^j : \Omega^{p,q} \to \Omega^{p+1,q+1}$, $\mathrm{tr} : \Omega^{p,q} \to \Omega^{p-1,q-1}$
- Partial Hodge stars: $*(\widetilde{*}) : \Omega^{p,q} \to \Omega^{D-p,q}(\Omega^{p,D-q})$ defined by

$$* \,\omega_{p,q} := \frac{1}{p!(D-p)!q!}\, \epsilon^{i_1 \dots i_p}{}_{i_{p+1} \dots i_D}\, \omega_{i_1 \dots i_p | j_1 \dots j_q}\, \theta^{i_{p+1}} \dots \theta^{i_D} \chi^{j_1} \dots \chi^{j_q}$$

$$\widetilde{*} \,\omega_{p,q} := \frac{1}{p!(D-q)!q!}\, \epsilon^{j_1 \cdots j_q}{}_{j_{q+1} \dots j_D}\, \omega_{i_1 \dots i_p | j_1 \dots j_q}\, \theta^{i_1} \dots \theta^{i_p} \chi^{j_{q+1}} \dots \chi^{j_D}$$

- Transposition: $\top : \Omega^{p,q} \to \Omega^{q,p}$ by $(\omega^\top)_{q,p} = \frac{1}{p!q!} \omega_{i_1 \dots i_p j_1 \dots j_q} \theta^{j_1} \dots \theta^{j_q} \chi^{i_1} \dots \chi^{i_p}$
- Co-differentials:

$$d^\dagger = (-1)^{1+D(p+1)} * d * : \Omega^{p,q} \to \Omega^{p-1,q}$$

$$\widetilde{d}^\dagger = (-1)^{1+D(q+1)}\, \widetilde{*}\, \widetilde{d}\, \widetilde{*} : \Omega^{p,q} \to \Omega^{p,q-1}$$

- Co-traces:

$$\sigma = (-1)^{1+D(p+1)} * \mathrm{tr} * : \Omega^{p,q} \to \Omega^{p+1,q-1}$$

$$\widetilde{\sigma} = (-1)^{1+D(q+1)}\, \widetilde{*}\, \mathrm{tr}\, \widetilde{*} : \Omega^{p,q} \to \Omega^{p-1,q+1}$$

Bipartite tensor representations of GL(D,$\mathbb{R}$)

- The space $\Omega^{p,q}(M)$ contains all GL(D,$\mathbb{R}$)-reducible bipartite tensors of type (p,q) .
- Given any bipartite tensor $\omega_{p,q} \in \Omega^{p,q}(M)$ there exists a unique bipartite tensor of the same type, say $\omega_{[p,q]}$, that satisfies the GL(D,$\mathbb{R}$)-irreducibility conditions

$$\begin{cases} \sigma\,\omega_{[p,q]} = 0 & \text{for} \quad p \geq q \\ \\ \widetilde{\sigma}\,\omega_{[p,q]} = 0 & \text{for} \quad p \leq q \end{cases}$$

$$\& \qquad \omega_{[p,q]} = \widetilde{\omega}_{[q,p]} \qquad \text{for} \qquad p = q$$

- The GL(D,$\mathbb{R}$)-irreducible tensor $\omega_{[p,q]}$ lives in a subspace $\Omega^{[p,q]}(M) \subseteq \Omega^{p,q}(M)$ and can be obtained by acting on $\omega_{p,q}$ with the Young Symmetrizer $\mathcal{P}_{[p,q]}$ given by [de Medeiros '04]

$$\mathcal{P}_{[p,q]} = \begin{cases} \mathbb{I} + \sum_{n=1}^{q} c_n(p,q)\,\sigma^n \widetilde{\sigma}^n & \text{for} \quad p \geq q \\ \\ \mathbb{I} + \sum_{n=1}^{p} c_n(q,p)\,\widetilde{\sigma}^n \sigma^n & \text{for} \quad p \leq q \end{cases}$$

A suitable Hodge star and kinetic terms

- We can also define a Hodge star operator $\star : \Omega^{p,q} \to \Omega^{D-p,D-q}$ by

$$\star\,\omega_{p,q} := \frac{1}{(D-p-q)!}\,\eta^{D-p-q}\,(\omega^{\top})_{q,p}$$

[Chatzistavrakidis, Khoo, Roest, Schupp '17], which is related to $*\,\widetilde{*}\,\omega_{p,q}$ via

$$\star\,\omega_{p,q} = (-1)^{(D-1)(p+q)+pq+1}\,*\,\widetilde{*}\sum_{n=0}^{\min(p,q)} \frac{(-1)^n}{(n!)^2}\,\eta^n\,\mathrm{tr}^n\,\omega_{p,q}$$

- The standard gauge-invariant kinetic term for any irreducible $\omega_{[p,q]}$ is then

$$\mathcal{L}_{\mathrm{kin}}(p,q) = \int_{\theta,\chi} d\omega_{[p,q]} \star d\omega_{[p,q]}\,, \qquad p+q+1 \leq D$$

where Berezin integration is used over θ and χ.

- Examples: $\mathcal{L}_{\mathrm{kin}}(0,0) = \mathcal{L}_{\mathrm{scalar}}$, $\mathcal{L}_{\mathrm{kin}}(1,0) = \mathcal{L}_{\mathrm{Maxwell}}$ and $\mathcal{L}_{\mathrm{kin}}(1,1) = \mathcal{L}_{\mathrm{LEH}}$.

E/M duality and types of duals

Electric/Magnetic duality relates two different free gauge theories in flat spacetime, which correspond to the same physical theory after full gauge fixing. Here we focus in Abelian theories.
In addition, for any gauge field there are different types of duals. For our purposes, we will consider the following:

- **Standard duals:** These are obtained by dualizing the original field in one of its sets of antisymmetrized indices. E.g. a p-form field only has one standard dual, while a bipartite tensor field has two. Some examples are the 1-form dual of a 1-form in $D = 4$ (self-duality) and the $[2, 1]$ tensor (Curtright field) dual of the $[1, 1]$ tensor (linearized graviton) in $D = 5$.
- **Double duals:** These duals do not exist for p-form fields. For bipartite tensors, they correspond to dualizing both its form sectors. As an example, the double dual graviton in $D = 5$ is a $[2, 2]$ tensor.
- **Exotic duals:** These are dual fields with additional sets of antisymmetrized indices compared to the original field, each one of length $(D - 2)$. Two examples are the $[2, 1]$ and the $[2, 2]$ tensor duals of a 1-form and a 2-form, respectively, in $D = 4$.

[Hull, Henneaux, West, Bergshoeff, Sundell, Boulanger, ...]

The universal parent Lagrangian

The universal parent Lagrangian we propose, in $D \geq p+q+1$, reads as

$$\mathcal{L}_{\mathrm{P}}^{(p,q)}(F_{p,q}, \lambda_{p+1,q}) = \int_{\theta,\chi} F \star \mathcal{O} F + \int_{\theta,\chi} dF * \widetilde{*}\, \lambda$$

- Depends on the 2 parameters $\{p, q\}$.
- Involves two independent reducible bipartite tensors F and λ.
- Contains an operator $\mathcal{O}^{(p,q)}$, defined by $\mathcal{O}^{(p,q)} d\omega_{p-1,q} \stackrel{!}{=} d\omega_{[p-1,q]} + \widetilde{d}(\dots)$. Its closed form contains the $\sigma(\widetilde{\sigma})$ maps:

$$\mathcal{O}^{(p,q)} = \begin{cases} \mathbb{I} + \sum_{n=1}^{q} c_n(p-1,q)\, \widetilde{\sigma}^n\, \sigma^n\,, & p \geq q+1 \\ \mathbb{I} + \sum_{n=1}^{p-1} c_n(q,p-1) \left(\sigma^n\, \widetilde{\sigma}^n + \sum_{k=1}^{n} (-1)^k \prod_{m=0}^{k-1} (n-m)^2 \sigma^{n-k}\, \widetilde{\sigma}^{n-k} \right)\,, & p < q+1 \end{cases}$$

Dualization procedure: Step 1

- Varying $\mathcal{L}_{\text{P}}$ w.r.t. $\lambda_{p+1,q}$ will give the Bianchi identity on $F_{p,q}$, namely $dF_{p,q} = 0$. Locally, this is solved by $F_{p,q} = d\omega_{p-1,q}$. Note that $\omega_{p-1,q}$ is GL-reducible.
- Substituting $F_{p,q} = d\omega_{p-1,q}$ back into $\mathcal{L}_{\text{P}}$ gives

$$\mathcal{L}^{(p,q)}_{\text{P},\lambda\text{-on-shell}} = \int_{\theta,\chi} d\omega_{p-1,q} \star d\omega_{[p-1,q]} + \text{s.t.}\,,$$

due to the action of $\mathcal{O}$ on the second F.
- One can prove that $\mathcal{L}^{(p,q)}_{\text{P},\lambda\text{-on-shell}}$ is truly the kinetic term for the GL-irreducible field $\omega_{[p-1,q]}$, i.e.

$$\mathcal{L}^{(p,q)}_{\text{P},\lambda\text{-on-shell}} = \int_{\theta,\chi} d\omega_{[p-1,q]} \star d\omega_{[p-1,q]} + \text{s.t.} = \mathcal{L}_{\text{kin}}(\omega_{[p-1,q]})\,,$$

if and only if the parameters $\{p,q\}$ take values in one of the four domains:

1. **Domain I:** $\{p \in [1, D-1]\,,\ q = 0\}$: S.D. of a $(p-1)$-form
2. **Domain II:** $\{p \in [2, D-2]\,,\ q = 1\}$: S.D. of a $[p-1, 1]$ bipartite tensor
3. **Domain III:** $\{p = 1\,,\ q \in [1, D-2]\}$: E.D. of a q-form
4. **Domain IV:** $\{p = 2\,,\ q \in [2, D-3]\}$: D.D. of a $[1, D-q-2]$ bipartite tensor

Dualization procedure: Step 2

- Alternatively, varying $\mathcal{L}_{\mathrm{P}}$ w.r.t. $F_{p,q}$ will give a Hodge duality relation of the form

$$\star\,\mathcal{O}^{(p,q)}F_{p,q} \sim \frac{1}{2}\,\widetilde{*}\,d*\lambda_{p+1,q}\,.$$

  Solving in terms of $F_{p,q}$ and substituting back into $\mathcal{L}_{\mathrm{P}}$ gives rise to a dual Lagrangian depending only on $\lambda_{p+1,q}$, i.e. $\mathcal{L}^{(p,q)}_{\mathrm{P},F\text{-on-shell}}(\lambda_{p+1,q})$.

- Decomposing $\lambda_{p+1,q}$ into its traceless and trace parts

$$\lambda_{p+1,q} \equiv \widehat{\lambda}_{p+1,q} + \eta\mathring{\lambda}_{p,q-1}\,, \qquad \mathrm{tr}\widehat{\lambda}_{p+1,q} \overset{!}{=} 0$$

  one can identify the GL-irreducible field $\widehat{\omega}_{[D-p-1,q]} := *\widehat{\lambda}_{p+1,q}$ with the E/M dual of $\omega_{[p-1,q]}$.

- However, not in every case does the extra 'trace' field $\mathring{\lambda}_{p,q-1}$ vanish from the dual Lagrangian. This only happens if $\{p\in[1,D-1]\,,\ q=0\}$ or $\{p\in[2,D-2]\,,\ q=1\}$, i.e. in the first and second domains of applicability of the parent Lagrangian, in which case the dual Lagrangian becomes the kinetic term for the dual field:

$$\mathcal{L}^{(p,q)}_{\mathrm{P},F\text{-on-shell}} = \int_{\theta,\chi} d\widehat{\omega}_{[D-p-1,q]}\star d\widehat{\omega}_{[D-p-1,q]} + \text{s.t.} = \mathcal{L}_{\mathrm{kin}}(\widehat{\omega}_{[D-p-1,q]})$$

Dualization procedure: Step 2

- For the remaining domains $\{p=1\,,\,q\in[1,D-2]\}$ and $\{p=2\,,\,q\in[2,D-3]\}$, the $\mathring{\lambda}_{p,q-1}$-dependence in the dual Lagrangian remains. However, the duality is manifest at the level of the equations of motion.
- **Example:** Consider the exotic dualization of a 1-form field, for which we use the parent Lagrangian $\mathcal{L}_{\mathrm{P}}^{(1,1)}$. Varying the dual Lagrangian w.r.t. the reducible field $\lambda_{2,1}$ gives the e.o.m.

$$d * d\,\widehat{\omega}_{[D-2,1]} - dd^{\dagger}\eta\mathring{\lambda}_{1,0} + \frac{1}{D-1}\,d\eta\,\mathrm{tr}\,d^{\dagger}\eta\mathring{\lambda}_{1,0} = 0 \qquad \Rightarrow \qquad \mathrm{tr}^2 d\widetilde{d}\,\widehat{\omega}_{[D-2,1]} = 0$$

  These equations are $\binom{D}{D-3}$ in number, while after full g.f. they are $\binom{D-2}{D-3} = D-2$. Not so surprisingly, the equations of motion for a 1-form field after full g.f. are also $D-2$ in number. Thus, the original and the dual Lagrangians imply the same number of propagating physical d.o.f.
- A similar argument holds also in the last domain $\{p=2\,,\,q\in[2,D-3]\}$.
- For any field $\zeta_{[p-1,q-1]}$ one can define the Riemann-like tensor $R_{[p,q]}(\zeta) := d\widetilde{d}\,\zeta$. An e.o.m. of the form $\mathrm{tr}^n R_{[p,q]} = 0$ would imply that $R_{[p,q]} = 0$ for $D < p+q+1-n$ [Hull '01]. Thus, one has to pose weaker e.o.m. In the above example $p = D-1$ and $q = 2$, so the first e.o.m. with non-trivial solutions is $\mathrm{tr}^2 R_{[D-1,2]}(\widehat{\omega}) = 0$.

Examples - Maxwell field and linearized graviton

- **S.D. of Maxwell field:** We set $\{p = 2, q = 0\}$ (Domain I) and use the parent Lagrangian $\mathcal{L}_{\mathrm{P}}^{(2,0)}$. The previous dualization procedure will relate the Maxwell 1-form field with its standard dual $(D-3)$-form field.
- **S.D. of linearized graviton:** We set $\{p = 2, q = 1\}$ (Domain II) and use the parent Lagrangian $\mathcal{L}_{\mathrm{P}}^{(2,1)}$. This relates the linearized graviton with its standard dual $[D-3, 1]$ bipartite tensor field.
- **E.D. of Maxwell field:** We set $\{p = 1, q = 1\}$ (Domain III) and use the parent Lagrangian $\mathcal{L}_{\mathrm{P}}^{(1,1)}$. At the level of e.o.m., we then get the duality between the Maxwell field and its exotic dual $\widehat{A}_{[D-2,1]}$:

$$d * d\, A_{1,0} = 0 = \mathrm{tr}^2\, R_{[D-1,2]}(\widehat{A})$$

- **D.D. of linearized graviton:** We set $\{p = 2, q = D-3\}$ (Domain IV) and use the parent $\mathcal{L}_{\mathrm{P}}^{(2,D-3)}$. Again, we have a duality between the linearized graviton and its double dual $\widehat{h}_{[D-3,D-3]}$:

$$d \star d\, h_{[1,1]} = 0 = \mathrm{tr}^{D-3}\, R_{[D-2,D-2]}(\widehat{h})$$

Conclusion and Outlook

We have constructed a 2-parameter Lagrangian $\mathcal{L}_{\mathrm{P}}^{(p,q)}$ capable of dualizing:

1. p-form fields into their standard duals (**Domain I**)
2. $[p, 1]$ bipartite tensors into their standard duals (**Domain II**)
3. p-form fields into their exotic duals (**Domain III**)
4. $[p, 1]$ bipartite tensors into their double duals (**Domain IV**)

and, thus, achieved all of our initial goals.

Future directions:

- Generalize for multipartite mixed-symmetry tensors. The formalism is obtained in a straightforward way by considering $\mathcal{M} = T[1]M \oplus \cdots \oplus T[1]M$.
- Study in more detail the exotic duality. For example, realize the infinite chain of dualities of [Boulanger, Sundell, West '15] in this geometric setting.
- Use this formalism to construct Galileon interactions for bipartite tensors [Chatzistavrakidis, Khoo, Roest, Schupp '17]. Multipartite tensor Galileons?

THANK YOU