Universidad de Burgos
Mathematical Physics Group

The κ-(A)dS noncommutative spacetime

Angel Ballesteros
I. Gutiérrez-Sagredo, F.J. Herranz

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'Noncommutative' space-times can be used to model (perhaps schematically) **quantum gravity features of spacetime** arising at the Planck scale.

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- For instance, q-deformed Casimir operators of quantum algebras generate **deformed dispersion relations** (DSR approach). The κ-Poincaré Casimir

$$\mathcal{C} = \frac{4}{z^2} \sinh^2(zP_0/2) - e^{zP_0} \mathbf{P}^2, \qquad q = e^z, \qquad z = 1/\kappa.$$

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- **Noncommutative (A)dS spacetimes** would be relevant to model **cosmological features** of such quantum gravity models.

The κ-Minkowski noncommutative spacetime

Probably the most relevant example of noncommutative spacetime with quantum group invariance is the **κ-Minkowski spacetime**

$$[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\,\hat{x}^a, \qquad [\hat{x}^a, \hat{x}^b] = 0, \qquad a, b = 1, 2, 3,$$

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- This noncommutative spacetime is covariant under the **κ-Poincaré quantum group** dual to the κ-Poincaré quantum algebra [Lukierski, Nowicki, Ruegg, Tolstoi 1991].
- Its 'semiclassical limit' is the **κ-Poincaré Poisson-Lie group** generated by the r-matrix
$$r = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3).$$
- The κ-Minkowski spacetime is just the **quantization of the Poisson homogeneous κ-Minkowski spacetime** arising from such r-matrix.

The κ-Minkowski noncommutative spacetime

κ-Minkowski spacetime as a benchmark for models aiming to describe different features of **quantum geometry at the Planck scale**:

- Noncommutative differential calculi and star products.
- Wave propagation on noncommutative spacetimes.
- Deformed Special Relativity features.
- Relative locality phenomena.
- Curved momentum spaces and phase spaces.
- Deformed dispersion relations.
- Noncommutative field theory.
- Representation theory.
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However, the **generalization of this type of noncommutative spacetime to the (A)dS case** with **nonvanishing Λ** was lacking.

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- **Plurality: for a given M there exist many possible QHS** coming from different quantum groups G_q that deform the same G.

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- Imposing that **H_q is a quantum subgroup** (*i.e.*, a Hopf subalgebra) turns out to be **too restrictive**: enough with H_q coisotropic [Dijkhuizen and Koornwinder, 1994].
- We adopt a **semiclassical approach** based on **Poisson homogeneous spaces**.

Poisson homogeneous spaces

Quantum groups G_q are **quantization of Poisson-Lie groups** (G, Π).

[1] See A.B., C. Meusburger, P. Naranjo, JPA (2017); arXiv:1612.03169.

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- The **'plurality'** for PHS/QHS can be examined at this level.
- In order to have a well-defined PHS the isotropy subgroup H has to be of **coisotropic type with respect to** δ: [1]

$$\delta(h) \subset h \wedge g.$$

Then the PHS bracket is just the **canonical projection onto** M **of the Sklyanin** bracket Π onto G.

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1. **Construct the classical (A)dS spaces** as the homogeneous spaces

$$\textbf{AdS}^{3+1} \equiv \mathrm{SO}(3,2)/\mathrm{SO}(3,1) \qquad \textbf{dS}^{3+1} \equiv \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$$

   and define appropriate coordinates on them.

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2. **Identify the κ-(A)dS classical r-matrix** r_Λ under two assumptions:
   a) The P_0 generator has to be primitive: $\delta(P_0) = 0$.
   b) The $\Lambda \to 0$ limit of r_Λ has to be the κ-Poincaré r-matrix.

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4. **Quantize the Poisson algebra** defined by the κ-(A)dS Poisson homogeneous space.

Main results

The only (A)dS classical r-matrix fulfilling the two previous conditions is

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2), \qquad \eta^2 = -\Lambda.$$

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The **quantization of the κ-(A)dS PHS up to first order in η** gives **time-space brackets** which are the same as in the κ-Minkowski spacetime

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while the **space-space subalgebra is noncommutative** and given by

$$[\hat{x}^1, \hat{x}^2] = -\frac{\eta}{\kappa}\,(\hat{x}^3)^2, \qquad [\hat{x}^1, \hat{x}^3] = \frac{\eta}{\kappa}\,\hat{x}^3\hat{x}^2, \qquad [\hat{x}^2, \hat{x}^3] = -\frac{\eta}{\kappa}\,\hat{x}^1\hat{x}^3,$$

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The noncommuting κ-(A)dS space coordinates generate "quantum spheres":

$$\hat{S}_{\eta/\kappa} = (\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 + \frac{\eta}{\kappa}\,\hat{x}^1\hat{x}^2, \qquad [S_{\eta/\kappa}, \hat{x}^i] = 0.$$

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Summary

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1. **Introduction**
2. **The κ-Minkowski noncommutative spacetime**

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Minkowski spacetime as an homogeneous space

The **(3+1)D Poincaré Lie algebra**:

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\begin{aligned}
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&[K_a, P_0] = P_a\,, &&[K_a, P_b] = \delta_{ab} P_0\,, &&[K_a, K_b] = -\epsilon_{abc} J_c\,,\\
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Casimir operators:

- A quadratic one

$$\mathcal{C} = P_0^2 - \mathbf{P}^2$$

- and the quartic one (Pauli-Lubanski)

$$\mathcal{W} = W_0^2 - \mathbf{W}^2$$

$$W_0 = \mathbf{J} \cdot \mathbf{P} \qquad W_a = -J_a P_0 + \epsilon_{abc} K_b P_c$$

The κ-Minkowski noncommutative spacetime

- **Faithful representation** ρ of the Poincaré Lie algebra:

$$\rho(X) = x^\alpha \rho(P_\alpha) + \xi^a \rho(K_a) + \theta^a \rho(J_a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\ x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & \xi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}.$$

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- An **element of the Poincaré group** $G = P(3+1)$ is parametrized in the form

$$G_{\mathcal{M}} = \exp\left(x^0 \rho(P_0)\right) \exp\left(x^1 \rho(P_1)\right) \exp\left(x^2 \rho(P_2)\right) \exp\left(x^3 \rho(P_3)\right)$$
$$\times \exp\left(\xi^1 \rho(K_1)\right) \exp\left(\xi^2 \rho(K_2)\right) \exp\left(\xi^3 \rho(K_3)\right) \exp\left(\theta^1 \rho(J_1)\right) \exp\left(\theta^2 \rho(J_2)\right) \exp\left(\theta^3 \rho(J_3)\right).$$

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- Hence the **Poincaré group matrix representation** takes the standard form

$$G_{\mathcal{M}} = \begin{pmatrix} 1 & \bar{0} \\ \bar{x}^T & \mathbf{\Lambda} \end{pmatrix},$$

where $\mathbf{\Lambda}$ is the 4×4 matrix representation of an element of the Lorentz subgroup.

The κ-Minkowski PHS

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\delta(P_0) &= \delta(J_a) = 0, \\
\delta(P_a) &= \frac{1}{\kappa} P_a \wedge P_0, \\
\delta(K_1) &= \frac{1}{\kappa}(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2), \\
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- The **coisotropy condition** is fulfilled: $h = \{J_1, J_2, J_3, K_1, K_2, K_3\}$ and

$$\delta(h) \subset h \wedge g\,.$$

The κ-Minkowski noncommutative spacetime

- The **Sklyanin bracket**

$$\{f,g\} = r^{ij}\left(X_i^L f\, X_j^L g - X_i^R f\, X_j^R g\right), \qquad f,g \in \mathcal{C}(G),$$

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- Its **canonical projection** to the Minkowski space M^{3+1} provides the PHS

$$\{x^0, x^a\} = -\frac{1}{\kappa}\, x^a, \qquad \{x^a, x^b\} = 0\,.$$

  where note that the **space coordinates Poisson commute**.

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$$\{x^0, x^a\} = -\frac{1}{\kappa}\, x^a, \qquad \{x^a, x^b\} = 0\,.$$

  where note that the **space coordinates Poisson commute**.

- This algebra is **covariant under the PL Poincaré (co)action**.

The κ-Minkowski noncommutative spacetime

- The **Sklyanin bracket**

$$\{f, g\} = r^{ij} \left(X_i^L f \, X_j^L g - X_i^R f \, X_j^R g \right), \qquad f, g \in \mathcal{C}(G),$$

  can be straightforwardly computed.

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  where note that the **space coordinates Poisson commute**.

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This linear bracket is **quantized** by replacing the Poisson brackets by commutators

$$[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\, \hat{x}^a, \qquad [\hat{x}^a, \hat{x}^b] = 0\,,$$

hence $\hat{x}^\alpha$ are the ‘noncommutative coordinates’ of the (κ-Minkowski) spacetime.

The κ-Poincaré quantum algebra

- The κ-Minkowski spacetime can be obtained through **Hopf algebra duality** from the κ-Poincaré Hopf algebra, with **deformed commutation rules**:

$$[K_a, P_b] = \delta_{ab}\left(\frac{\kappa}{2}\left(1 - \mathrm{e}^{-2P_0/\kappa}\right) + \frac{1}{2\kappa}\,\mathbf{P}^2\right) - \frac{1}{\kappa}\,P_aP_b$$

and **deformed coproduct** given by

$$\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta(J_a) &= J_a \otimes 1 + 1 \otimes J_a, \\
\Delta(P_a) &= P_a \otimes 1 + \mathrm{e}^{-P_0/\kappa} \otimes P_a, \\
\Delta(K_a) &= K_a \otimes 1 + \mathrm{e}^{-P_0/\kappa} \otimes K_a + \frac{1}{\kappa}\,\epsilon_{abc} P_b \otimes J_c\,.
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- Quantum Minkowski coordinates are **dual generators to the deformed translations**

$$\langle \hat{x}^\alpha, P_\beta \rangle = \delta^\alpha_\beta.$$

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- The Lie bialgebra is the first order of the coproduct deformation.

3. The κ-(A)dS Poisson homogeneous spacetime

The (A)dS algebra

The **(3+1)D (A)dS Lie algebra**:

$$
\begin{aligned}
&[J_a, J_b] = \epsilon_{abc} J_c\,, && [J_a, P_b] = \epsilon_{abc} P_c\,, && [J_a, K_b] = \epsilon_{abc} K_c\,,\\
&[K_a, P_0] = P_a\,, && [K_a, P_b] = \delta_{ab} P_0\,, && [K_a, K_b] = -\epsilon_{abc} J_c\,,\\
&[P_0, P_a] = -\Lambda\, K_a\,, && [P_a, P_b] = \Lambda\, \epsilon_{abc} J_c\,, && [P_0, J_a] = 0\,.
\end{aligned}
$$

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\end{array}
$$

In this way we have a common presentation of:

- $\Lambda < 0$: AdS spacetime $\mathbf{AdS}^{3+1} \equiv \mathrm{SO}(3,2)/\mathrm{SO}(3,1)$.
- $\Lambda > 0$: dS spacetime $\mathbf{dS}^{3+1} \equiv \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$.
- $\Lambda = 0$: Minkowski spacetime $\mathbf{M}^{3+1} \equiv \mathrm{ISO}(3,1)/\mathrm{SO}(3,1)$.

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[J_a, J_b] = \epsilon_{abc} J_c\,, & [J_a, P_b] = \epsilon_{abc} P_c\,, & [J_a, K_b] = \epsilon_{abc} K_c\,, \\
[K_a, P_0] = P_a\,, & [K_a, P_b] = \delta_{ab} P_0\,, & [K_a, K_b] = -\epsilon_{abc} J_c\,, \\
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Casimir operators: a quadratic one

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 - \Lambda \left(\mathbf{J}^2 - \mathbf{K}^2\right)$$

and the quartic one (Pauli-Lubanski)

$$\mathcal{W} = W_0^2 - \mathbf{W}^2 - \Lambda\,(\mathbf{J}\cdot\mathbf{K})^2$$

$$W_0 = \mathbf{J}\cdot\mathbf{P} \qquad W_a = -J_a P_0 + \epsilon_{abc} K_b P_c$$

(3+1) Lorentzian homogeneous spaces

$$h = \{J_1, J_2, J_3, K_1, K_2, K_3\}$$

Λ	Spacetimes
$\Lambda < 0$	$\mathbf{AdS}^{3+1} = SO(3,2)/SO(3,1)$
$\Lambda = 0$	$\mathbf{M}^{3+1} = ISO(3,1)/SO(3,1)$
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The (A)dS group and its parametrizations

- **Faithful representation** ρ of the (A)dS Lie algebra:

$$\rho(X) = x^0\rho(P_0)+x^a\rho(P_a)+\xi^a\rho(K_a)+\theta^a\rho(J_a) = \begin{pmatrix} 0 & \Lambda x^0 & -\Lambda x^1 & -\Lambda x^2 & -\Lambda x^3 \\ x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\ x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & \xi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}.$$

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- The **(A)dS group element is parametrized** in the form

$$G_\Lambda = \exp\left(x^0\rho(P_0)\right)\exp\left(x^1\rho(P_1)\right)\exp\left(x^2\rho(P_2)\right)\exp\left(x^3\rho(P_3)\right)$$
$$\times\exp\left(\xi^1\rho(K_1)\right)\exp\left(\xi^2\rho(K_2)\right)\exp\left(\xi^3\rho(K_3)\right)\exp\left(\theta^1\rho(J_1)\right)\exp\left(\theta^2\rho(J_2)\right)\exp\left(\theta^3\rho(J_3)\right).$$

$$G_\Lambda = \begin{pmatrix} s^4 & A^4_0 & A^4_1 & A^4_2 & A^4_3 \\ s^0 & B^0_0 & B^0_1 & B^0_2 & B^0_3 \\ s^1 & B^1_0 & B^1_1 & B^1_2 & B^1_3 \\ s^2 & B^2_0 & B^2_1 & B^2_2 & B^2_3 \\ s^3 & B^3_0 & B^3_1 & B^3_2 & B^3_3 \end{pmatrix},$$

where the entries A^α_β and B^μ_ν now depend on all the group coordinates $(x^0, \mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\theta})$ and on the cosmological constant Λ.

- **'Ambient' coordinates** for the (A)dS spacetime

$$(s^4, s^0, \mathbf{s})^T = \exp x^0 \rho(P_0) \exp x^1 \rho(P_1) \exp x^2 \rho(P_2) \exp x^3 \rho(P_3) \cdot O^T.$$

with $O = (1, 0, 0, 0, 0)$, are related with 'local' ones through:

$$s^4 = \cos \eta x^0 \cosh \eta x^1 \cosh \eta x^2 \cosh \eta x^3,$$
$$s^0 = \frac{\sin \eta x^0}{\eta} \cosh \eta x^1 \cosh \eta x^2 \cosh \eta x^3,$$
$$s^1 = \frac{\sinh \eta x^1}{\eta} \cosh \eta x^2 \cosh \eta x^3,$$
$$s^2 = \frac{\sinh \eta x^2}{\eta} \cosh \eta x^3,$$
$$s^3 = \frac{\sinh \eta x^3}{\eta},$$

where the parameter η is defined by $\eta^2 := -\Lambda$.

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\end{aligned}$$

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- The (3+1)-dimensional **(A)dS spacetime is defined by the pseudosphere**

$$\Sigma_\Lambda \equiv (s^4)^2 - \Lambda (s^0)^2 + \Lambda \left((s^1)^2 + (s^2)^2 + (s^3)^2 \right) = 1.$$

Minkowski spacetime is identified with the hyperplane $s^4 = +1$.

The κ-(A)dS r-matrix

Task: **identify the κ-(A)dS classical r-matrix** r_Λ under two assumptions,
a) The P_0 **generator** has to be **primitive:** $\delta(P_0) = 0$.
b) The $\Lambda \to 0$ **limit** of r_Λ has to be the **κ-Poincaré r-matrix**.

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- Take a generic **skewsymmetric pre-r-matrix** depending on 45 parameters.
- Impose onto it the **mCYBE and conditions a) and b)**. We get:

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + P_0 \wedge (\beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3) + \alpha_3 J_1 \wedge J_2 - \alpha_2 J_1 \wedge J_3 + \alpha_1 J_2 \wedge J_3 \,,$$

$$\beta_1\alpha_3 - \beta_3\alpha_1 = 0, \qquad \beta_1\alpha_2 - \beta_2\alpha_1 = 0, \qquad \beta_2\alpha_3 - \beta_3\alpha_2 = 0,$$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \left(\frac{\eta}{\kappa}\right)^2 .$$

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- The β_i terms are twists and only the α_3 term is left under automorphisms.

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$$\beta_1\alpha_3 - \beta_3\alpha_1 = 0, \qquad \beta_1\alpha_2 - \beta_2\alpha_1 = 0, \qquad \beta_2\alpha_3 - \beta_3\alpha_2 = 0,$$
$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \left(\frac{\eta}{\kappa}\right)^2 .$$

- The β_i terms are twists and only the α_3 term is left under automorphisms.
- Then the **unique solution** is

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2) \,.$$

The (3+1) κ-(A)dS Lie bialgebra

The **classical r-matrix** is

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2)\,.$$

The (3+1) κ-(A)dS Lie bialgebra

The **classical *r*-matrix** is

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2)\,.$$

and the **cocommutator map** reads (it is again coisotropic: $\delta(h) \subset h \wedge g$)

$$\delta(P_0) = \delta(J_3) = 0, \qquad \delta(J_1) = \frac{\eta}{\kappa} J_1 \wedge J_3, \qquad \delta(J_2) = \frac{\eta}{\kappa} J_2 \wedge J_3,$$

$$\delta(P_1) = \frac{1}{\kappa}(P_1 \wedge P_0 - \eta P_3 \wedge J_1 - \eta^2 K_2 \wedge J_3 + \eta^2 K_3 \wedge J_2),$$

$$\delta(P_2) = \frac{1}{\kappa}(P_2 \wedge P_0 - \eta P_3 \wedge J_2 + \eta^2 K_1 \wedge J_3 - \eta^2 K_3 \wedge J_1),$$

$$\delta(P_3) = \frac{1}{\kappa}(P_3 \wedge P_0 + \eta P_1 \wedge J_1 + \eta P_2 \wedge J_2 - \eta^2 K_1 \wedge J_2 + \eta^2 K_2 \wedge J_1),$$

$$\delta(K_1) = \frac{1}{\kappa}(K_1 \wedge P_0 + P_2 \wedge J_3 - P_3 \wedge J_2 - \eta K_3 \wedge J_1),$$

$$\delta(K_2) = \frac{1}{\kappa}(K_2 \wedge P_0 - P_1 \wedge J_3 + P_3 \wedge J_1 - \eta K_3 \wedge J_2),$$

$$\delta(K_3) = \frac{1}{\kappa}(K_3 \wedge P_0 + P_1 \wedge J_2 - P_2 \wedge J_1 + \eta K_1 \wedge J_1 + \eta K_2 \wedge J_2).$$

(3+1) κ-(A)dS Poisson homogeneous space

By **computing the Sklyanin bracket** for the κ-(A)dS r-matrix ($\eta^2 = -\Lambda$) we get

$$\{x^0, x^1\} = -\frac{1}{\kappa}\,\frac{\tanh(\eta x^1)}{\eta\cosh^2(\eta x^2)\cosh^2(\eta x^3)},$$

$$\{x^0, x^2\} = -\frac{1}{\kappa}\,\frac{\tanh(\eta x^2)}{\eta\cosh^2(\eta x^3)},$$

$$\{x^0, x^3\} = -\frac{1}{\kappa}\,\frac{\tanh(\eta x^3)}{\eta},$$

$$\{x^1, x^2\} = -\frac{1}{\kappa}\,\frac{\cosh(\eta x^1)\tanh^2(\eta x^3)}{\eta},$$

$$\{x^1, x^3\} = \frac{1}{\kappa}\,\frac{\cosh(\eta x^1)\tanh(\eta x^2)\tanh(\eta x^3)}{\eta},$$

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$$\{x^2, x^3\} = -\frac{1}{\kappa} \frac{\sinh(\eta x^1) \tanh(\eta x^3)}{\eta},$$

The κ-Minkowski PHS $\mathbf{M}_z^{3+1}$ is obtained in the $\eta \to 0$ limit.

(3+1) κ-(A)dS Poisson homogeneous space

If we take the zeroth-order expansion in terms of η we get the **κ-Minkowski PHS**

$$\{x^0, x^1\} = -\frac{1}{\kappa}\,(x^1 + o[\eta^2]),$$
$$\{x^0, x^2\} = -\frac{1}{\kappa}\,(x^2 + o[\eta^2]),$$
$$\{x^0, x^3\} = -\frac{1}{\kappa}\,(x^3 + o[\eta^2]),$$

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and the **first-order deformation in η** of the space subalgebra defines an **homogeneous quadratic algebra** of noncommutative translation parameters

$$\{x^1, x^2\} = -\frac{1}{\kappa}\,(\eta\,(x^3)^2 + o[\eta^2]),$$
$$\{x^1, x^3\} = \frac{1}{\kappa}\,(\eta\,x^2 x^3 + o[\eta^2]),$$
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$$\{x^2, x^3\} = -\frac{1}{\kappa}\,(\eta\,x^1 x^3 + o[\eta^2]).$$

- **Symplectic leaves** for the latter structure are **spheres**

$$S = (x^1)^2 + (x^2)^2 + (x^3)^2.$$

4. The κ-(A)dS noncommutative spacetime

(3+1) κ-(A)dS noncommutative spacetime

The **quantization of the κ-(A)dS PHS up to first order in η** can be obtained by considering the basis of ordered monomials

$$(\hat{x}^1)^l\,(\hat{x}^3)^m\,(\hat{x}^2)^n\,.$$

The **time-space brackets** are the same as in the κ-Minkowski spacetime

$$[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\,\hat{x}^a, \qquad a = 1, 2, 3$$

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while the **space-space subalgebra** is given by

$$[\hat{x}^1,\hat{x}^2] = -\frac{\eta}{\kappa}\,(\hat{x}^3)^2, \qquad [\hat{x}^1,\hat{x}^3] = \frac{\eta}{\kappa}\,\hat{x}^3\hat{x}^2, \qquad [\hat{x}^2,\hat{x}^3] = -\frac{\eta}{\kappa}\,\hat{x}^1\hat{x}^3,$$

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$$[\hat{x}^0,\hat{x}^a] = -\frac{1}{\kappa}\,\hat{x}^a, \qquad a = 1,2,3$$

while the **space-space subalgebra** is given by

$$[\hat{x}^1,\hat{x}^2] = -\frac{\eta}{\kappa}\,(\hat{x}^3)^2, \qquad [\hat{x}^1,\hat{x}^3] = \frac{\eta}{\kappa}\,\hat{x}^3\hat{x}^2, \qquad [\hat{x}^2,\hat{x}^3] = -\frac{\eta}{\kappa}\,\hat{x}^1\hat{x}^3,$$

- The **Casimir operator for the space subalgebra** is

$$\hat{S}_{\eta/\kappa} = (\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 + \frac{\eta}{\kappa}\,\hat{x}^1\hat{x}^2,$$

and defines the **'quantum spheres'** generated by the noncommuting κ-(A)dS local coordinates.

(3+1) κ-(A)dS noncommutative spacetime

For all orders in η we consider the PHS in **ambient coordinates**:

$$\{s^0, s^a\} = -\frac{1}{\kappa}\, s^a s^4, \qquad \{s^4, s^a\} = \frac{\eta^2}{\kappa}\, s^a s^0, \qquad a = 1, 2, 3,$$

$$\{s^1, s^2\} = -\frac{\eta}{\kappa}(s^3)^2, \qquad \{s^1, s^3\} = \frac{\eta}{\kappa}\, s^2 s^3, \qquad \{s^2, s^3\} = -\frac{\eta}{\kappa}\, s^1 s^3,$$

$$\{s^0, s^4\} = -\frac{\eta^2}{\kappa}\left((s^1)^2 + (s^2)^2 + (s^3)^2\right),$$

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and its **quantization** in the basis $\{(\hat{s}^0)^k\,(\hat{s}^1)^l\,(\hat{s}^3)^m\,(\hat{s}^2)^n\,(\hat{s}^4)^j\}$ is given by

$$[\hat{s}^0, \hat{s}^a] = -\frac{1}{\kappa}\,\hat{s}^a\hat{s}^4, \qquad [\hat{s}^4, \hat{s}^a] = \frac{\eta^2}{\kappa}\,\hat{s}^0\hat{s}^a, \qquad [\hat{s}^0, \hat{s}^4] = -\frac{\eta^2}{\kappa}\,\hat{\mathcal{S}}_{\eta/\kappa},$$
$$[\hat{s}^1, \hat{s}^2] = -\frac{\eta}{\kappa}(\hat{s}^3)^2, \qquad [\hat{s}^1, \hat{s}^3] = \frac{\eta}{\kappa}\,\hat{s}^3\hat{s}^2, \qquad [\hat{s}^2, \hat{s}^3] = -\frac{\eta}{\kappa}\,\hat{s}^1\hat{s}^3,$$

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Here $\hat{\mathcal{S}}_{\eta/\kappa}$ is the analogue of the **quantum sphere** in quantum ambient coordinates

$$\hat{\mathcal{S}}_{\eta/\kappa} = (\hat{s}^1)^2 + (\hat{s}^2)^2 + (\hat{s}^3)^2 + \frac{\eta}{\kappa}\,\hat{s}^1\hat{s}^2\,,$$

and the Casimir operator gives the **κ-(A)dS space** as the **'quantum pseudosphere'**

$$\hat{\Sigma}_{\eta,\kappa} = (\hat{s}^4)^2 + \eta^2(\hat{s}^0)^2 - \frac{\eta^2}{\kappa}\,\hat{s}^0\hat{s}^4 - \eta^2\hat{\mathcal{S}}_{\eta/\kappa}.$$

5. Concluding remarks

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- The Poisson analogue of the full **quantum κ-(A)dS algebra** has been obtained: [2]

$$\omega^2 = \eta^2 = -\Lambda$$

$$\begin{aligned}
\Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\
\Delta(J_1) &= J_1 \otimes e^{z\sqrt{\omega}J_3} + 1 \otimes J_1, \qquad \Delta(J_2) = J_2 \otimes e^{z\sqrt{\omega}J_3} + 1 \otimes J_2,
\end{aligned}$$

$$\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta(P_1) &= P_1 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes P_1 - \sqrt{\omega}K_2 \otimes \sinh(z\sqrt{\omega}J_3) \\
&\quad -z\sqrt{\omega}P_3 \otimes J_1 + z\omega K_3 \otimes J_2 + z^2\omega\left(\sqrt{\omega}K_1 - P_2\right) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3} \\
&\quad -\frac{1}{2}z^2\omega\left(\sqrt{\omega}K_2 + P_1\right) \otimes \left(J_1^2 - J_2^2\right) e^{-z\sqrt{\omega}J_3}, \\
\Delta(P_2) &= P_2 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes P_2 + \sqrt{\omega}K_1 \otimes \sinh(z\sqrt{\omega}J_3) \\
&\quad -z\sqrt{\omega}P_3 \otimes J_2 - z\omega K_3 \otimes J_1 - z^2\omega\left(\sqrt{\omega}K_2 + P_1\right) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3} \\
&\quad -\frac{1}{2}z^2\omega\left(\sqrt{\omega}K_1 - P_2\right) \otimes \left(J_1^2 - J_2^2\right) e^{-z\sqrt{\omega}J_3}, \\
\Delta(P_3) &= P_3 \otimes 1 + e^{-zP_0} \otimes P_3 + z\left(\omega K_2 + \sqrt{\omega}P_1\right) \otimes J_1 e^{-z\sqrt{\omega}J_3} \\
&\quad -z\left(\omega K_1 - \sqrt{\omega}P_2\right) \otimes J_2 e^{-z\sqrt{\omega}J_3},
\end{aligned}$$

[2] A.B. , F.J. Herranz, F. Musso, P. Naranjo, PLB (2017), arXiv:1612.03169.

Concluding remarks

$$
\begin{aligned}
\Delta(K_1) &= K_1 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes K_1 + P_2 \otimes \frac{\sinh(z\sqrt{\omega}J_3)}{\sqrt{\omega}} \\
&\quad -zP_3 \otimes J_2 - z\sqrt{\omega}K_3 \otimes J_1 - z^2\left(\omega K_2 + \sqrt{\omega}P_1\right) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3} \\
&\quad -\frac{1}{2}z^2\left(\omega K_1 - \sqrt{\omega}P_2\right) \otimes \left(J_1^2 - J_2^2\right) e^{-z\sqrt{\omega}J_3}, \\
\Delta(K_2) &= K_2 \otimes \cosh(z\sqrt{\omega}J_3) + e^{-zP_0} \otimes K_2 - P_1 \otimes \frac{\sinh(z\sqrt{\omega}J_3)}{\sqrt{\omega}} \\
&\quad +zP_3 \otimes J_1 - z\sqrt{\omega}K_3 \otimes J_2 - z^2\left(\omega K_1 - \sqrt{\omega}P_2\right) \otimes J_1 J_2 e^{-z\sqrt{\omega}J_3} \\
&\quad +\frac{1}{2}z^2\left(\omega K_2 + \sqrt{\omega}P_1\right) \otimes \left(J_1^2 - J_2^2\right) e^{-z\sqrt{\omega}J_3}, \\
\Delta(K_3) &= K_3 \otimes 1 + e^{-zP_0} \otimes K_3 + z(\sqrt{\omega}K_1 - P_2) \otimes J_1 e^{-z\sqrt{\omega}J_3} \\
&\quad +z(\sqrt{\omega}K_2 + P_1) \otimes J_2 e^{-z\sqrt{\omega}J_3}.
\end{aligned}
$$

The $so_q(3)$ **sub-Hopf algebra** characteristic of the Drinfel'd-Jimbo deformation arises:

$$
\{J_1, J_2\} = \frac{e^{2z\sqrt{\omega}J_3} - 1}{2z\sqrt{\omega}} - \frac{z\sqrt{\omega}}{2}\left(J_1^2 + J_2^2\right), \qquad \{J_1, J_3\} = -J_2, \qquad \{J_2, J_3\} = J_1.
$$

Concluding remarks

- In **(2+1) dimensions** the κ-(A)dS r-matrix is

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2),$$

and the κ-(A)dS spacetime has commuting space translations

$$[x_1, x_2] = 0.$$

[3] A.B., I. Gutiérrez-Sagredo, F.J. Herranz, PLB (2019); arXiv:1902.09132.

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- This approach **works for any other coisotropic quantum deformation of Lorentzian groups**.

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- This approach **works for any other coisotropic quantum deformation of Lorentzian groups**.
- **Noncommutative time-like spaces of worldlines** can be constructed: [3]

$$h = \{J_1, J_2, J_3, P_0\}$$

Λ	6D spaces of time-like worldlines
$\Lambda < 0$	$\mathbf{LAdS}^{3+1} = SO(3,2)/\left(SO(3) \otimes SO(2)\right)$
$\Lambda = 0$	$\mathbf{LM}^{3+1} = ISO(3,1)/\left(SO(3) \otimes \mathbb{R}\right)$
$\Lambda > 0$	$\mathbf{LdS}^{3+1} = SO(4,1)/\left(SO(3) \otimes SO(2,1)\right)$

[3] A.B., I. Gutiérrez-Sagredo, F.J. Herranz, PLB (2019); arXiv:1902.09132.

THANKS FOR YOUR ATTENTION!