3D/4D Gravity
as Dimensional Reduction of 3-forms in 6D/7D

Based on work with K.Krasnov (UoN) and C.Scarinci (KIAS)

Yannick Herfray (Université Libre de Bruxelles)

ENS DE LYON

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Overview

In this talk, I will present an unusual perspective on 3D/4D ‘gravity’.

The essential idea is that ‘gravity theories’ can be obtained from a theory of a completely different type (Hitchin Theory) in higher dimension 6D/7D.

Overview

In this talk, I will present an unusual perspective on 3D/4D ‘gravity’.

The essential idea is that ‘gravity theories’ can be obtained from a theory of a completely different type (Hitchin Theory) in higher dimension 6D/7D.

The building blocks I will use (Stable Forms, Hitchin’s functional) are closely tied up with the early history of Generalised Geometry

Overview

In this talk, I will present an unusual perspective on 3D/4D ‘gravity’.

The essential idea is that ‘gravity theories’ can be obtained from a theory of a completely different type (Hitchin Theory) in higher dimension 6D/7D.

The building blocks I will use (Stable Forms, Hitchin’s functional) are closely tied up with the early history of Generalised Geometry

I wish I could provide a full ‘Generalised’ version of these results today, but this is still ongoing work... Stay tuned !

Stable Forms and Hitchin's Volumes

It has to do with the Linear Algebra of k-skew-linear forms $\Lambda^k(\mathbb{R}^n)$

$\Rightarrow$ results are more than 100 years old !

W.Reichel, Dissertation (1907),

Uber die Trilinearen Alternierenden Formen in 6 und 7 Veranderlichen

Geometry of Stable Forms

It has to do with the Linear Algebra of k-skew-linear forms $\Lambda^k(\mathbb{R}^n)$

$\Rightarrow$ results are more than 100 years old !

W.Reichel, Dissertation (1907),
Uber die Trilinearen Alternierenden Formen in 6 und 7 Veranderlichen

$\Rightarrow$ re-clarified/advertised by N.Hitchin in

"*The geometry of three-forms in six and seven dimensions*",
J.Diff.Geom, (2000)

Geometry of Stable Forms

It has to do with the Linear Algebra of k-skew-linear forms $\Lambda^k(\mathbb{R}^n)$

$\Rightarrow$ results are more than 100 years old !

W.Reichel, Dissertation (1907),
Uber die Trilinearen Alternierenden Formen in 6 und 7 Veranderlichen

$\Rightarrow$ re-clarified/advertised by N.Hitchin in

"*The geometry of three-forms in six and seven dimensions*",
J.Diff.Geom, (2000)

... and ultimately led to
"*Generalised Calabi-Yau Manifolds*", N.Hitchin, Quart.J.Math (2003)

Stable Forms: Definition

Starting points: $\mathrm{GL}(n, \mathbb{R})$ acts on $\Lambda^k(\mathbb{R}^n)$.

Let $\Omega \in \Lambda^k(\mathbb{R}^n)$ and $g \in \mathrm{GL}(n, \mathbb{R})$, then

$$g.\Omega(X_1, ..., X_k) \coloneqq \Omega(g(X_1), ..., g(X_k))$$

Stable Forms: Definition

Starting points: $\mathrm{GL}(n, \mathbb{R})$ acts on $\Lambda^k(\mathbb{R}^n)$.

Let $\Omega \in \Lambda^k(\mathbb{R}^n)$ and $g \in \mathrm{GL}(n, \mathbb{R})$, then

$$g.\Omega\left(X_1, ..., X_k\right) := \Omega\left(g\left(X_1\right), ..., g\left(X_k\right)\right)$$

One can then consider the orbit $\mathcal{O}_\Omega$ of a k-form $\Omega \in \Lambda^k(\mathbb{R}^n)$.

Definition: 'Stable k-Form' (Hitchin)

$\Omega \in \Lambda^k(\mathbb{R}^n)$ is called *stable*

if $\mathcal{O}_\Omega$ is an open subset of $\Lambda^k(\mathbb{R}^n)$

Stable Forms: Why is it a useful definition ?

Stable Forms: Why is it a useful definition ?

• Fact 1 : In any even dimension $n = 2m$, stable 2-forms are equivalent to non-degenerate 2-forms.

$B \in \Omega^2(M^{2m})$ is stable $\Leftrightarrow det(B_{\mu\nu}) \neq 0$

Thus *stability* generalises 'non-degeneracy' to any k-form !

Stable Forms: Why is it a useful definition ?

- Fact 1 : In any even dimension $n = 2m$, stable 2-forms are equivalent to non-degenerate 2-forms.

$$B \in \Omega^2(M^{2m}) \text{ is stable} \Leftrightarrow det(B_{\mu\nu}) \neq 0$$

Thus *stability* generalises 'non-degeneracy' to any k-form !

- Fact 2 : Out of stable k-forms one can construct a -$\mathrm{GL}(n)$ equivariant- volume functional,

'Hitchin's volume-form'

$$\Phi : \Omega^k_{stable}(M^n) \to \Omega^n_{stable}(M^n) \quad (1)$$

$\Rightarrow$ for $B \in \Omega^2(M^{2m})$ this is just the Liouville volume $\Phi\,(B) := B^m$

Stable Forms: A rare phenomenon

Here is the complete list:

- stable 0 and 1-forms are *non-zero* 0 and 1-forms
- stable 2-forms are *maximally non-degenerate* 2-forms
- stable 3-forms *only exist in 6,7 and 8 dimensions*

(NB: any-time k-forms can be stable, $n-k$ -forms can be stable.)

(some of) Hitchin's main results in 6D

Let Ω be a stable 3-form on an 6- manifold M.

(some of) Hitchin's main results in 6D

Let Ω be a stable 3-form on an 6- manifold M.

Theorem (Hitchin, 2000):

Then $\Omega = \alpha^1 \wedge \alpha^2 \wedge \alpha^3 + \tilde{\alpha}^1 \wedge \tilde{\alpha}^2 \wedge \tilde{\alpha}^3$. Their are two possible orbits ('positive or negative'), depending on the reality condition

> We suppose that Ω is in the negative orbit, then $\tilde{\alpha} = \overline{\alpha}$ and it defines an almost complex structure J_Ω.

and
Ω is a critical point on a cohomology class of the 6D Hitchin's Volume if and only if

$$J_\Omega \text{ is integrable}$$

Hitchin's 6D functional and 3D gravity

Dimensional Reduction

Dimensional reduction:

- we suppose that $SU(2)$ has a free action on M^6
- we suppose that Ω is left invariant by this action
- we then consider the resulting theory on the quotient space $M^3 = M^6/SU(2)$

Dimensional Reduction

Dimensional reduction:

- we suppose that $SU(2)$ has a free action on M^6
- we suppose that Ω is left invariant by this action
- we then consider the resulting theory on the quotient space $M^3 = M^6/SU(2)$

Practically, one

- reads-off the field content of (the stable, SU(2)-invariant) 3-form Ω,
- imposes the constraint $d\Omega = 0$,
- evaluates the 6D Hitchin's functional on Ω,
- gives a 3D interpretation to the resulting functional and field equations.

Take home message

The dimensional reduction of the 6D Hitchin functional by $SU(2)$ gives

3D gravity with non-zero cosmological constant

(together with a constant scalar field).

What is more,

the *sign of the cosmological constant* then correspond to
the sign of the orbit of the stable 3-form Ω.

Take home message

The dimensional reduction of the 6D Hitchin functional by $SU(2)$ gives

3D gravity with non-zero cosmological constant

(together with a constant scalar field).

What is more,

the *sign of the cosmological constant* then correspond to *the sign of the orbit* of the stable 3-form Ω.

> The dimensionally reduced Hitchin functional is then proportional to the *pure connection action* of 3D gravity.

Metric versus Connection action for 3D gravity

$$S_P\left[g,\omega\right]=\int_{M^3}F^i\left(\omega\right)\wedge e^i\ +\Lambda e^i\wedge e^j\wedge e^k\frac{\epsilon_{ijk}}{6}$$

Two Possible Perspectives:

Metric versus Connection action for 3D gravity

$$S_P\left[g,\omega\right] = \int_{M^3} F^i\left(\omega\right)\wedge e^i \; + \Lambda e^i \wedge e^j \wedge e^k \frac{\epsilon_{ijk}}{6}$$

Two Possible Perspectives:

'Metric Perspective'

- $\frac{\partial S}{\partial \omega} = 0$
- $\omega = fct\left(g\right)$
- $S_{EH}\left[g\right] = S_P\left[g, \omega\left(g\right)\right]$
- Einstein equations: $\frac{\partial S}{\partial g} = 0$

Einstein's Equations are equations on g

Metric versus Connection action for 3D gravity

$$S_P\left[g,\omega\right] = \int_{M^3} F^i\left(\omega\right)\wedge e^i \; + \Lambda e^i \wedge e^j \wedge e^k \frac{\epsilon_{ijk}}{6}$$

Two Possible Perspectives:

‘Metric Perspective’

- $\frac{\partial S}{\partial \omega} = 0$
- $\omega = fct\left(g\right)$
- $S_{EH}\left[g\right] = S_P\left[g, \omega\left(g\right)\right]$
- Einstein equations: $\frac{\partial S}{\partial g} = 0$

Einstein’s Equations are equations on g

‘Connection Perspective’

- $\frac{\partial S}{\partial g} = 0$
- $g = fct\left(\omega\right)$
- $S_{PC}\left[\omega\right] = S_P\left[g\left(\omega\right), \omega\right]$
- Einstein equations: $\frac{\partial S}{\partial \omega} = 0$

Einstein’s Equations are equations on ω

Take home message

The dimensional reduction of the 6D Hitchin functional by $SU(2)$ gives

3D gravity with non-zero cosmological constant

(together with a constant scalar field).

What is more,

the *sign of the cosmological constant* then correspond to *the sign of the orbit* of the stable 3-form Ω.

The dimensionally reduced Hitchin functional is then proportional to the *pure connection action* of 3D gravity.

Why does it work so well here?

3D gravity has a natural 6D interpretation in terms of Cartan geometry:

- Take a 3d Riemannian manifold (M^3, g) and consider its bundle of frame $P^6 \simeq SU(2) \times M^3$,
- When g is a solution to 3D gravity then (at least locally)

$$M^3 = SL(2,\mathbb{C})/SU(2) \quad \text{and} \quad P^6 \simeq SL(2,\mathbb{C}),$$

- In general $P^6 \neq SL(2,\mathbb{C})$ but we have a Cartan connection

$$A^i = (g^{-1}dg)^i + \left(g^{-1}\,(e + i\omega)\,g\right)^i \qquad \in \Gamma\left[T^* M^6\right] \times \mathfrak{su}2$$

which makes the infinitesimal identification,

$$TP^6 \to \mathfrak{sl}(2,\mathbb{C})$$

- The vanishing of its curvature is then equivalent to 3D gravity.

Generalised version of these results?

Stable Spinors

As we know,
differential forms in n dimensions can be thought as spinors S for $SO(n,n)$.

This suggest the following generalisation (Hitchin 2003) :

Stable Forms were defined as forms such that their orbits under $GL(n) \subset SO(n,n)$ is open in $\Lambda^k(M^n)$.

Stable Spinors are thus defined as spinors such that their orbit under $SO(n,n)$ is open in S

One can then cook up ‘generalised Hitchin functionals’ out of the natural inner products on S. Cf N.Hitchin (2003) and F.Witt (2005.)

6D Case

Let Ω be a stable *Spinor* on an 6- manifold M.

Theorem (Hitchin, 2003):

Then $\Omega = \alpha + \tilde{\alpha}$ with α and β two *complex pure spinors*. Their are again two possible orbits ('positive or negative').

We suppose that Ω is in the negative orbit, then $\tilde{\alpha} = \overline{\alpha}$ and it defines a generalised complex structure $V_\Omega \subset (T \oplus T^*) \otimes \mathbb{C}$

finally
Ω is a critical point on a cohomology class of the 6D Hitchin's functional if and only if

$$V_\Omega \text{ is (Courant)-integrable}$$

Together with Ω this gives a generalised Calabi-Yau manifold.

Open questions

- An elegant embedding of 3D gravity into 6D Hitchin theory through a clean dimensional reduction.

  How does our dimensional reduction generalise?
  What is this ‘generalised 3D gravity’ we should obtain?

- Intriguingly, from 7D, the dimensional reduction of Hitchin theory gives ‘gravity theories’ but, as it stands, this is messy and inconclusive.

  In 7D, Stable spinors give rise to Generalised G_2-manifold.
  Dimensional reduction of this structure should shed some new light on the above result.