Renormalization of the Yang–Mills spectral action

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Spectral action in noncommutative geometry

Geometrical description of several theories in high-energy physics ...

such as:

- Yang–Mills theory and the Standard Model of elementary particle physics, including Higgs.
- Supersymmetric Yang–Mills (Thijs van den Broek)

Technique: asymptotically expand the spectral action $\mathrm{Tr}(f(D/\Lambda))$ to obtain (physical, but Euclidean) Lagrangians at lowest order in the fields.

... at the classical level

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This turns out to be surprisingly successful:

- In the case of the Yang–Mills system, the spectral action, regularized by Λ defines a higher-derivative gauge theory which is superrenormalizable. [vS, arXiv:1101.4804,1104.5199]
- More generally, for AC manifolds $M \times F$ one can define conditions on the finite space F such that the corresponding spectral action (with Λ) is superrenormalizable (though not multiplicatively). [vS,in preparation]

The spectral action for Yang–Mills fields

- Starting point is the Dirac operator $\partial\!\!\!/ = i\gamma^\mu \partial_\mu$ on a flat (4-dimensional) Riemannian spin manifold M, coupled to a gauge field $A_\mu(x) \in \mathfrak{su}(N)$:

$$\partial\!\!\!/ \rightsquigarrow \partial\!\!\!/_A := i\gamma^\mu(\partial_\mu + A_\mu)$$

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- Gauge transformations are given by

$$A_\mu \mapsto uA_\mu u^* + u\partial_\mu u^*; \qquad (u(x) \in SU(N)).$$

In other words, $\not\partial_A \mapsto u\not\partial_A u^*$.

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- A natural gauge invariant functional is given by the spectral action [CC]

$$\boxed{S_\Lambda[A] = \operatorname{Tr} f(\partial\!\!\!/_A/\Lambda) - \operatorname{Tr} f(\partial\!\!\!/ /\Lambda).}$$

  where $f : \mathbb{R} \to \mathbb{R}_+$ is an arbitrary even function that we assume to be a Laplace transform

$$f(x) = \int_{t>0} e^{-tx^2} g(t)dt.$$

Yang–Mills action at lowest order

Theorem (Chamseddine–Connes 1996)

The spectral action for the above Yang–Mills system is given, asymptotically as $\Lambda \to \infty$, by

$$S_\Lambda[A] \sim -\frac{f(0)}{24\pi^2} \int_M \mathrm{Tr}_N\, F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(\Lambda^{-1})$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

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The appearance of the Yang–Mills action at lowest order is the main motivation to study this model: we will soon incorporate the terms $\propto \Lambda^{-1}$.

Crashcourse: renormalization of the Yang–Mills action

$$
\begin{aligned}
S_{\mathrm{YM}}[A] &= -\frac{1}{4}\,\mathrm{Tr}_N\, F_{\mu\nu}F^{\mu\nu} \\
&= -\frac{1}{2}\,\mathrm{Tr}_N\, \partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}\partial_\nu A_\mu[A^\mu, A^\nu] - \frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu]
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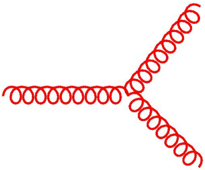
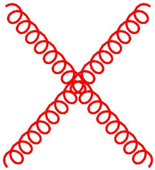
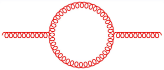
- The interaction terms of order 3 and 4 gives rise to vertices

 $\sim |p|$;  ~ 1 (as $p \to \infty$)

These edges and vertices connect to form *Feynman graphs*, such as

 $\propto \int d^4p \frac{p_\mu p_\nu}{p^2(q+p)^2}, \quad \int d^4p \frac{p_\mu}{p^2(q+p)^2}, \quad \int d^4p \frac{1}{p^2(q+p)^2}$

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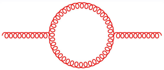

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More generally, for a Feynman graph Γ at loop order L, with I (E) internal (external) edges and V_3, V_4 vertices, the number of momenta in numerator and denominator is superficial degree of divergence

$$\omega = 4L - 2I + V_3 = 4 - E$$

If $\omega > 0$ then the amplitude is (potentially) divergent; if $\omega < 0$ then the amplitude is convergent.

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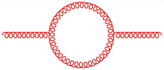

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Conclusion: For Yang–Mills theory the divergent Feynman graphs have $E \leq 4$ (any L): renormalizable field theory.

Counterterm is $\propto A, A^2, A^3, A^4$; gauge invariance $\Longrightarrow \mathrm{Tr}_N\, F_{\mu\nu} F^{\mu\nu}$

Full asymptotic expansion and higher-derivatives

Proposition (CC)

There is an asymptotic expansion (as $\Lambda \to \infty$):

$$S_\Lambda[A] \equiv \operatorname{Tr} f(\not{\partial}_A/\Lambda) - f(\not{\partial}/\Lambda) \sim \sum_{m>0} \Lambda^{4-m} f_{4-m} \int_M a_m(x, \not{\partial}_A^2).$$

where a_m are the heat kernel invariants of the Laplace-type operator $\not{\partial}_A^2$.

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- The $a_m(x, \not{\partial}_A^2)$ are Lorentz and gauge invariant polynomials of degree m in the covariant derivative $\partial_\mu + A_\mu$.

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- The $a_m(x, \partial\!\!\!/_A^2)$ are Lorentz and gauge invariant polynomials of degree m in the covariant derivative $\partial_\mu + A_\mu$.
- For example,

$$a_4(x, \partial\!\!\!/_A^2) = \frac{1}{8\pi^2}\left(-\frac{1}{3}\operatorname{Tr}_N F_{\mu\nu}F^{\mu\nu}\right),$$

$$a_6(x, \partial\!\!\!/_A^2) = \frac{1}{8\pi^2}\left(\frac{2}{15}\operatorname{Tr}_N F^{\mu\nu}{}_{;\mu}F^{\rho\nu}{}_{;\rho} + \frac{23}{45}\operatorname{Tr}_N F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu\right).$$

The free part

Proposition (vS)

The spectral action admits the following asymptotic expansion (as $\Lambda \to \infty$):

$$S_\Lambda[A] \sim -\int_M \mathrm{Tr}_N\, F_{\mu\nu} \varphi_\Lambda(\Delta)(F^{\mu\nu}) + \mathcal{O}(A^3)$$

where $\varphi_\Lambda(\Delta) = \sum_{k \geq 0} (-1)^k \Lambda^{-2k} f_{-2k} c_k \Delta^k$; $\qquad c_k = \frac{1}{8\pi^2} \frac{(k+1)!}{(2k+3)(2k+1)!}$.

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- Quadratic part vanishes for pure gauge fields $A_\mu = \partial_\mu \chi$; **gauge fixing:**

$$S_{\mathrm{gf}}[A] \sim -\frac{1}{2\xi} \int \mathrm{Tr}_N \, \partial_\mu A^\mu \varphi_\Lambda(\Delta) \, (\partial_\nu A^\nu)$$

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- This yields a **gauge propagator** of the form:

$$\boxed{D^{ab}_{\mu\nu}(p;\Lambda) = \left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]\frac{\delta^{ab}}{p^2\varphi_\Lambda(p^2)}}$$

Faddeev–Popov ghosts and Jacobian

- As usual, the above gauge fixing requires a Jacobian

$$S_{\mathrm{gh}}[A, C, \overline{C}] \sim -\int \mathrm{Tr}_N\, \partial_\mu \overline{C} \varphi_\Lambda(\Delta)\, (\partial^\mu C + [A^\mu, C])$$

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Proposition

The sum $S_\Lambda[A] + S_{\mathrm{gf}}[A] + S_{\mathrm{gh}}[A, C, \overline{C}]$ is invariant under the BRST-transformations:

$$sA_\mu = \partial_\mu C + [A_\mu, C]; \qquad sC = -\tfrac{1}{2}[C, C]; \qquad s\overline{C} = \xi^{-1}\partial_\mu A^\mu.$$

The parameter Λ as a regulator

We treat $S_\Lambda[A] + S_{\rm gf}[A] + S_{\rm gh}[A, C, \overline{C}]$ as a higher-derivative gauge theory, with Λ acting as a regulator.

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$$\varphi_\Lambda(p^2) = \sum_{k=0}^{n/2-2} \Lambda^{-2k} f_{-2k} c_k p^{2k} \sim p^{n-4} \quad (\text{as } |p| \to \infty).$$

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- This implies for the gauge and ghost propagator:

$$D^{ab}_{\mu\nu}(p;\Lambda), \quad \widetilde{D}^{ab}(p;\Lambda) \sim p^{-n+2} \quad (\text{as } |p| \to \infty).$$

Feynman graph for the HD theory

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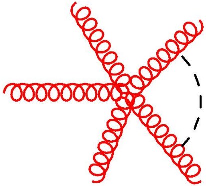

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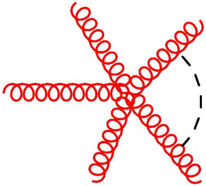

$$\sim |p|^{n-i} \qquad (\text{as } p \to \infty)$$

- This implies that for a Feynman graph at loop order L with E external edges, the superficial degree of divergence is

$$\boxed{\omega = (4-n)(L-1) + 4 - E}$$

Superrenormalizability

- Thus, with $\omega = (4 - n)(L - 1) + 4 - E$ we conclude that if $n \geq 8$ then
  - $\omega < 0$ if $L \geq 2$ (convergent)
  - $\omega > 0$ if $L = 1$ and $E \leq 4$ (finitely many divergent graphs, 1L)

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- The required counterterms all appear at one-loop, and have maximal degree $= 4$ in the fields and derivatives.
- BRST-invariance then implies that the counterterms are of the form

$$\delta Z \int_M \mathrm{Tr}_N \, F_{\mu\nu} F^{\mu\nu}.$$

which can be absorbed into the spectral action by translating

$$f(x) \mapsto f(x) + 24\pi^2 \delta Z$$

Conclusions (I)

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- This opens intriguing possibilities in trying to quantize this theory.
- First, we focus on the relation with noncommutative geometry and derive a general superrenormalizability result for certain almost commutative (AC) manifolds.

Noncommutative manifolds

- Basic device: a spectral triple $(\mathcal{A}, \mathcal{H}, D)$:
  - algebra $\mathcal{A}$ of bounded operators on
  - a Hilbert space $\mathcal{H}$,
  - a self-adjoint operator D with compact resolvent such that the commutator $[D, a]$ is bounded for all $a \in \mathcal{A}$.

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- Grading $\gamma : \mathcal{H} \to \mathcal{H}$ such that

$$\gamma^2 = \mathrm{id}, \qquad D\gamma + \gamma D = 0, \qquad \gamma a = a\gamma \quad (a \in A)$$

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- Real structure $J : \mathcal{H} \to \mathcal{H}$, anti-unitary operator such that

$$JD = \pm JD, \qquad J\gamma = \pm \gamma J.$$

defining an $\mathcal{A}$-bimodule structure on $\mathcal{H}$ via

$$(a, b) \cdot \psi = aJb^* J^{-1}\psi \quad (\psi \in \mathcal{H})$$

and we require (first order):

$$[[D, a], JbJ^{-1}] = 0$$

Example: Riemannian spin geometry

Let M be a compact m-dimensional Riemannian spin manifold.

- $\mathcal{A} = C^\infty(M)$
- $\mathcal{H} = L^2(S)$, square integrable spinors
- $D = \not{\partial}$, Dirac operator
- $\gamma = \gamma_{m+1}$ if m even (chirality)
- $J = C$ (charge conjugation)

Then D has compact resolvent because $\not{\partial}$ elliptic self-adjoint.
Also $[D, f]$ bounded for $f \in C^\infty(M)$ with $\|[D, f]\| = \|f\|_{\text{Lip}}$.

AC Manifolds

Let M be a compact m-dimensional Riemannian spin manifold, and $(A_F, H_F, D_F; \gamma_F, J_F)$ a spectral triple for which H_F is finite-dimensional.

- $\mathcal{A} = C^\infty(M, A_F)$
- $\mathcal{H} = L^2(S) \otimes H_F$, square integrable spinors
- $D = \partial\!\!\!/ \otimes 1 + \gamma_{m+1} \otimes D_F$, Dirac operator
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Two examples of interest:

- The algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices acting on itself

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- The noncommutative description of the Standard Model is based on

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

  It is represented on $H_F = \mathbb{C}^{96}$, where 96 is 2 (particles and anti-particles) times 3 (families) times 4 leptons plus 4 quarks with 3 colors each. Finally, there are 96×96 matrices D_F, γ_F, and J_F.

Classification of AC manifolds

Since

$$A_F \simeq \bigoplus_{i=1}^{N} M_{k_i}(\mathbb{C}) \qquad \text{(complex algebras)}$$

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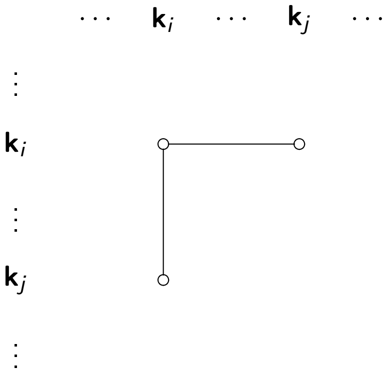
Graphically, presence of such an irrep in H_F can be depicted by a node:

	$\cdots$	$\mathbf{k}_i$	$\cdots$	$\mathbf{k}_j$	$\cdots$
$\vdots$					
$\mathbf{k}_i$		$\circ$		$\circ$	
$\vdots$					
$\mathbf{k}_j$		$\circ$			
$\vdots$					

- J_F is the reflection along the diagonal.
- The operator $D_F : H_F \to H_F$ is represented by lines between nodes;



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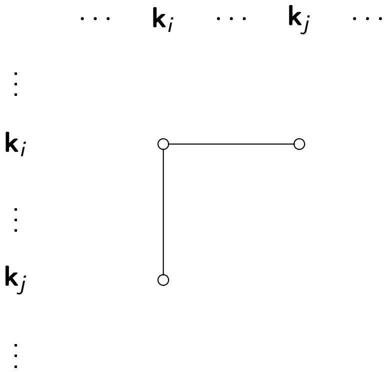
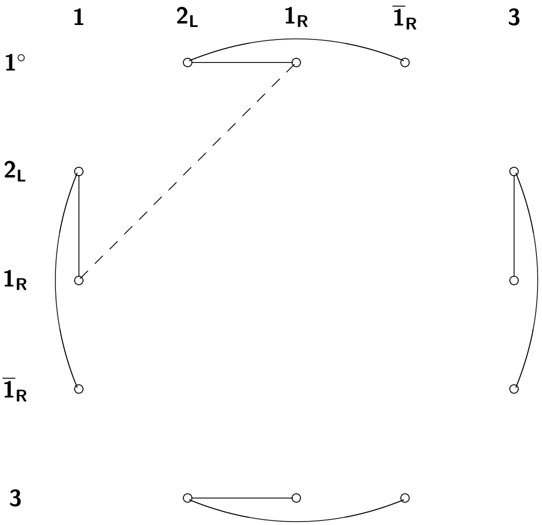


The conditions on a real spectral triple demand that the lines run horizontally or vertically and that the **Krajewski diagram** is symmetric with respect to the diagonal.

Krajewski diagram for the SM: $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$



Inner fluctuations for (general) spectral triples

- Inner fluctuations (driven by Morita self-equivalences of $\mathcal{A}$) replace the operator D by $D' = D_A \equiv D + A \pm JAJ^{-1}$ with $A^* = A \in \Omega^1_D(\mathcal{A})$ the connection one-form (gauge potential) in $\nabla = d + A$, where

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum_k a_k[D, b_k] : a_k, b_k \in \mathcal{A} \right\}$$

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- In this generality, a gauge invariant functional of A is given by the spectral action

$$S_\Lambda[A] := \mathrm{Tr}(f(D_A/\Lambda) - f(D/\Lambda))$$

with f a function on $\mathbb{R}$ (...) and $\Lambda \in \mathbb{R}$ a cutoff parameter.

Inner fluctuations for AC manifolds and the spectral action

- Inner fluctuations replace $\partial\!\!\!/ \rightsquigarrow \partial\!\!\!/ + iA + \gamma_5 \Phi$ where $A \in \Omega^1(M, \mathfrak{u}(A_F))$ and $\Phi^*(x) = \Phi(x) \in \Omega^1_{D_F}(A_F)$.

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Proposition (vS)

The spectral action for 4d AC manifolds admits an asympt. exp. ($\Lambda \to \infty$):

$$S_\Lambda[A, \Phi] \sim -\frac{f_2}{4\pi^2} \int_M \mathrm{Tr}_F\, \Phi^2 + \int_M \mathrm{Tr}_F(\partial_\mu \Phi) \vartheta_\Lambda(\Delta)(\partial^\mu \Phi) - \int \mathrm{Tr}_F\, F_{\mu\nu} \varphi_\Lambda(\Delta)(F^{\mu\nu}) + \mathcal{O}(A^3, \Phi^3)$$

with $\varphi_\Lambda, \vartheta_\Lambda$ formal expansions in Λ.

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- Similarly, a Faddeev–Popov ghost term, producing a BRST-invariant functional.

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- On the other hand, in the spectral action only particular invariant functional will appear, such as $\mathrm{Tr}_F\,\Phi^4$, $\mathrm{Tr}_F\,\Phi^2$, $\mathrm{Tr}_F(\nabla_\mu \Phi)^2$.

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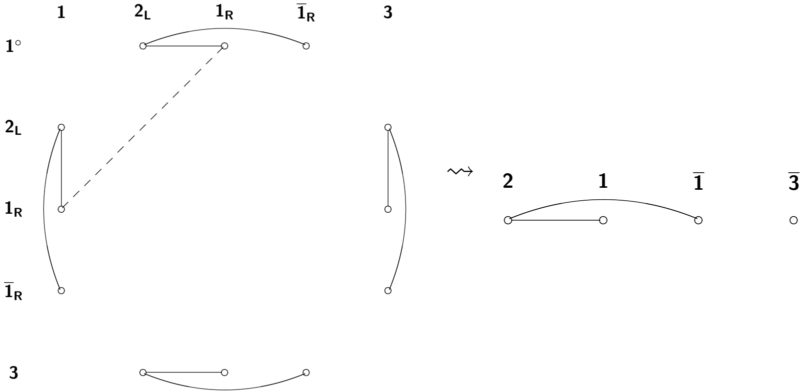
- On the other hand, the functionals in the components of Φ that appear in the spectral action correspond to cycles in Γ (idem).

Theorem (vS)

Consider an AC manifold given by a Krajewski diagram Γ. The corresponding spectral action, considered as a higher-derivative gauge theory with regulator Λ is superrenormalizable if:

every union of connected cycles in $\widetilde{\Gamma}$ of total length 4 can be lifted to a connected cycle in Γ.

Example: The Standard Model



Thus, the Krajewski diagram for the Standard Model satisfies this property, so that the corresponding spectral action is superrenormalizable as a higher-derivative gauge theory with regularizing parameter Λ.

Conclusions (II)

- We have established that under certain (graph theoretical) conditions, the spectral action (interpreted as HD-theory, Λ) on an AC manifold is superrenormalizable as a gauge theory.
- The spectral action becomes the Lagrangian of physical interest (eg. YM, SM), asymptotically as $\Lambda \to \infty$: this motivates the role of Λ as a regulator.
- This is in contrast with a recent preprint [ILV] where the full spectral action was considered, without such a regulator. In that case, the propagator has behaviour $\sim p^{-4}$.
- The renormalizability is not multiplicative, since typically (SM) the spectral action defines a theory at unification.
- The next step consists of combining the spectral action as HD regulator of the physical Lagrangian with eg. dim-reg to regularize the divergences at one loop and compute (and compare) the β-function in this scheme.