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U(2) family of BC's

S-matrix & bound state energy

Exact RG flow of BC's
- Exact β-functions
- Exact RG flow

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- $1/r^2$ potential
- Quantum graph

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Running Boundary Conditions in Quantum Mechanics

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INFN Sezione di Pisa

ERG 2010, September 12 – 19, Corfu, Greece

based on

SO, M. Sakamoto, M. Tachibana, arXiv:1005.4676[hep-th]

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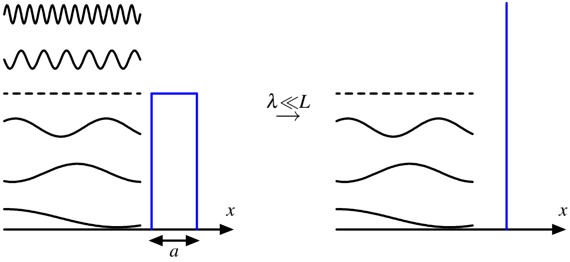
- Slowly moving particle cannot resolve the structure of a short-ranged scatterer (e.g. an impurity or a defect).
- (Much) below a physical cutoff scale a, any short-ranged interaction could be approximated by a point interaction.

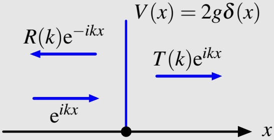


short-ranged interaction $\xrightarrow{\text{long-wavelength limit}}$ point interaction (boundary condition)

Question: Do there exist any universality classes of short-ranged interactions whose long-wavelength limits appear to be the same? (Yes. It is described by running boundary conditions.)

Introduction ②: trivial example

Flow of S-matrix elements



$$R(k;g) = \frac{g}{ik-g}$$

$$T(k;g) = \frac{ik}{ik-g}$$

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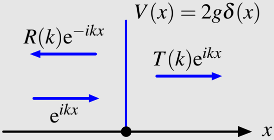

$$-1 \xleftarrow{k\to 0} R(k;g) = \frac{g}{ik-g}$$

$$0 \xleftarrow{k\to 0} T(k;g) = \frac{ik}{ik-g}$$

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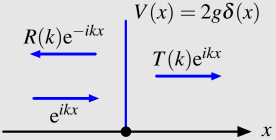

$$-1 \overset{k\to 0}{\longleftarrow} R(k;g) = \frac{g}{ik-g} \overset{k\to\infty}{\longrightarrow} 0$$

$$0 \overset{k\to 0}{\longleftarrow} T(k;g) = \frac{ik}{ik-g} \overset{k\to\infty}{\longrightarrow} 1$$

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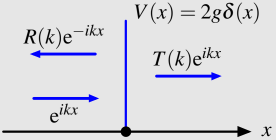

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Momentum rescaling $k \to k\mathrm{e}^t$ as running coupling

$$R(k\mathrm{e}^t;g) = \frac{g}{ik\mathrm{e}^t - g} = \frac{g\mathrm{e}^{-t}}{ik - g\mathrm{e}^{-t}} = R(k;\bar{g}(t))$$

$$T(k\mathrm{e}^t;g) = \frac{ik\mathrm{e}^t}{ik\mathrm{e}^t - g} = \frac{ik}{ik - g\mathrm{e}^{-t}} = T(k;\bar{g}(t))$$

with

$$\bar{g}(t) = g\mathrm{e}^{-t}, \quad -\infty < t < \infty$$

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Today's talk

- Spinless one-particle quantum mechanics on a line
- Absence of long-ranged interaction (such as Coulomb force)
- Long-wavelength limit

$\Rightarrow$ Theory space (spanned by parameters which characterize all possible point interactions)

||

Parameter space of $U(2)$

$U(2)$ family of boundary conditions ①

Allowed point interaction in quantum mechanics

||

Boundary condition consistent with the probability conservation
($\Leftrightarrow$ self-adjointness of H $\Leftrightarrow$ unitary time evolution e^{-itH})

Current conservation at the origin [cf. Cheon et al. (2000)]

$$j(0_+) = j(0_-), \quad j(x) := -i\left[\psi'^*(x)\psi(x) - \psi^*(x)\psi'(x)\right]$$

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$$\Leftrightarrow \left|\begin{pmatrix}\psi(0_+)\\ \psi(0_-)\end{pmatrix} - iL_0\begin{pmatrix}\psi'(0_+)\\ \psi'(0_-)\end{pmatrix}\right|^2 = \left|\begin{pmatrix}\psi(0_+)\\ \psi(0_-)\end{pmatrix} + iL_0\begin{pmatrix}\psi'(0_+)\\ \psi'(0_-)\end{pmatrix}\right|^2$$

(L_0: arbitrary length scale)

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(L_0: arbitrary length scale)

$$\Leftrightarrow \begin{pmatrix}\psi(0_+)\\ \psi(0_-)\end{pmatrix} - iL_0\begin{pmatrix}\psi'(0_+)\\ \psi'(0_-)\end{pmatrix} = U\left[\begin{pmatrix}\psi(0_+)\\ \psi(0_-)\end{pmatrix} + iL_0\begin{pmatrix}\psi'(0_+)\\ \psi'(0_-)\end{pmatrix}\right]$$

$U \in U(2)$

$U(2)$ family of boundary conditions ②

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- $U(2)$ family of boundary conditions

$$(\mathbb{1}-U)\begin{pmatrix}\psi(0_+)\\ \psi(0_-)\end{pmatrix} - iL_0(\mathbb{1}+U)\begin{pmatrix}\psi'(0_+)\\ \psi'(0_-)\end{pmatrix} = \vec{0}, \quad U \in U(2)$$

- Parameterization of $U \in U(2)$

$$U = \sum_{j=\pm} \mathrm{e}^{i\alpha_j} P_j, \quad P_\pm = \frac{\mathbb{1} \pm \vec{e}\cdot\vec{\sigma}}{2}$$

- 4 independent parameters

$$0 \le \alpha_+, \alpha_- < 2\pi,$$
$$\vec{e} = (e_x, e_y, e_z) \quad \text{with} \quad e_x^2 + e_y^2 + e_z^2 = 1$$

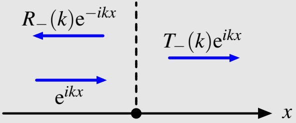
- $U(2)$ family of boundary conditions reduces to the following two independent conditions:

$$P_\pm\left[\begin{pmatrix}\psi(0_+)\\ \psi(0_-)\end{pmatrix} + L_\pm\begin{pmatrix}\psi'(0_+)\\ \psi'(0_-)\end{pmatrix}\right] = \vec{0}, \quad L_\pm := L_0 \cot\frac{\alpha_\pm}{2}$$

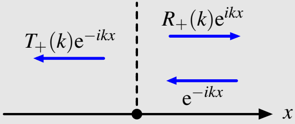
S-matrix

- Reflection and transmission coefficients $(k > 0)$

From left



From right



- S-matrix

$$S(k) = \begin{pmatrix} R_+ & T_- \\ T_+ & R_- \end{pmatrix} = \sum_{j=\pm} \frac{ikL_j - 1}{ikL_j + 1} P_j$$

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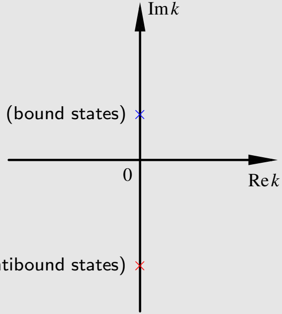
Bound states and antibound (or virtual) states

- Bound (antibound) state = simple pole of $S(k)$ lying on the **positive** (**negative**) imaginary k-axis
- Bound/antibound state energy at a pole $k = i/L_\pm$:

$$E_\pm = \left(\frac{i}{L_\pm}\right)^2 = -\frac{1}{L_\pm^2}$$



- Bound/antibound state wave function at a pole $k = i/L_\pm$:

$$\psi_\pm(x) \propto \exp\left(-\frac{|x|}{L_\pm}\right) \quad \left(L_\pm = L_0 \cot\frac{\alpha_\pm}{2}, \quad L_0 > 0\right)$$

$$\Rightarrow \begin{cases} \text{normalizable bound state} & \text{for } 0 < \alpha_\pm < \pi \\ \text{non-normalizable antibound state} & \text{for } \pi < \alpha_\pm < 2\pi \end{cases}$$

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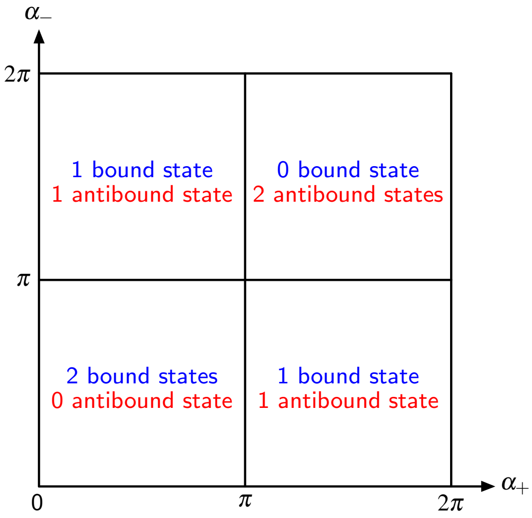

Exact RG flow of boundary conditions ①

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Renormalization group transformation

Since L_0 is an arbitrary parameter, any physical quantities must be independent of the choice of L_0. The lack of dependence of L_0 can be expressed as the invariance of the theory under the RG transformation

$$R_t : L_0 \mapsto \bar{L}(t) := L_0 \mathrm{e}^{-t}, \quad -\infty < t < \infty.$$

Renormalization group equation

Any change of L_0 must be equivalent to changes in the $U(2)$ parameters. This is expressed as the RG equation

$$S(kL_0; \alpha_j, P_j) = S(k\bar{L}(t); \bar{\alpha}_j(t), \bar{P}_j(t)),$$

or, equivalently,

$$S((k\mathrm{e}^t)L_0; \alpha_j, P_j) = S(kL_0; \bar{\alpha}_j(t), \bar{P}_j(t)).$$

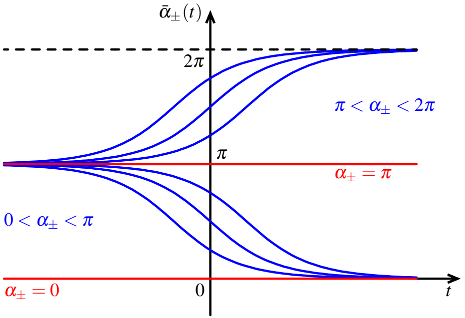
Exact RG flow of boundary conditions ②

Since k appears only in the combination $kL_\pm = kL_0 \cot\frac{\alpha_\pm}{2}$, the rescaling of k must be adjusted by the running of $\alpha_\pm$:

$$kL_0 \cot\frac{\alpha_\pm}{2} \overset{k\to k\mathrm{e}^t}{\longmapsto} kL_0\left(\mathrm{e}^t \cot\frac{\alpha_\pm}{2}\right) = kL_0\left(\cot\frac{\bar{\alpha}_\pm(t)}{2}\right),$$

from which we find

$$\bar{\alpha}_\pm(t) = 2\arctan\left(\mathrm{e}^{-t}\tan\frac{\alpha_\pm}{2}\right) \quad \text{and} \quad \bar{P}_\pm(t) = P_\pm.$$



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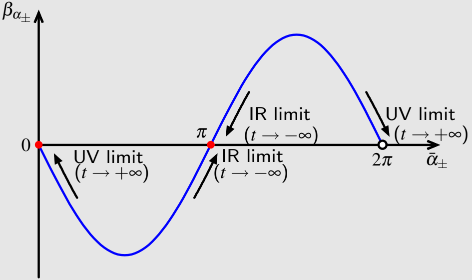
Exact β-functions

$$\beta_{\alpha_\pm}(\bar{\alpha}_\pm(t)) := \left.\frac{\partial\bar{\alpha}_\pm(t)}{\partial t}\right|_{\alpha_\pm, L_0} = -\sin\bar{\alpha}_\pm(t)$$



- $\alpha_\pm^* = 0$: UV fixed point
- $\alpha_\pm^* = \pi$: IR fixed point
- $2^2 = 4$ fixed points on $T^2 = \{(\alpha_+, \alpha_-) \mid 0 \le \alpha_\pm < 2\pi\}$

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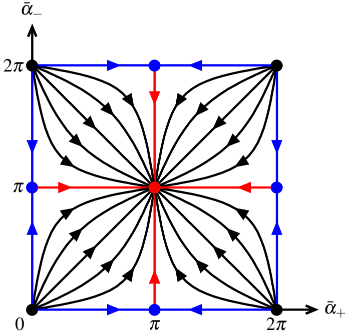

Arrows indicate the directions toward infrared.

- Neumann fixed point (UV stable)

$$U = S = \mathbb{1}$$
$$\psi'(0_+) = 0 = \psi'(0_-)$$

- Dirichlet fixed point (IR stable)

$$U = S = -\mathbb{1}$$
$$\psi(0_+) = 0 = \psi(0_-)$$

- $(0, \pi)$ fixed point

$$U = S = \begin{pmatrix} \cos\varphi & e^{-i\theta}\sin\varphi \\ e^{i\theta}\sin\varphi & -\cos\varphi \end{pmatrix}$$
$$\psi(0_-) = e^{i\theta}\tan\frac{\varphi}{2}\psi(0_+)$$
$$\psi'(0_-) = e^{i\theta}\cot\frac{\varphi}{2}\psi'(0_+)$$

Applications: $1/r^2$ potential

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- Hamiltonian (in one-dimension)

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{g}{r^2}, \quad 0 < r < \infty$$

- Short-distance behavior of wavefunction

$$\psi(r) = r^{\nu_+} \times (\text{regular polynomial}) + r^{\nu_-} \times (\text{regular polynomial})$$

where $\nu_\pm := 1/2 \pm \sqrt{g + 1/4}$.

- Boundary condition at $r = 0$ (for $-1/4 < g < 3/4$)

$$\left[\frac{\psi(r)}{r^{\nu_-}} + L_0^{\nu_+ - \nu_-}\left(\cos\frac{\alpha}{2}\right) r^{2\nu_-}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\psi(r)}{r^{\nu_-}}\right]_{r=0} = 0$$

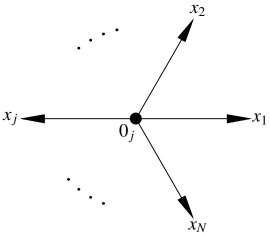
- RG equation

$$L_0^{\nu_+ - \nu_-}\cos\frac{\alpha}{2} = [\bar{L}(t)]^{\nu_+ - \nu_-}\cos\frac{\bar{\alpha}(t)}{2}$$

- Exact β-function

$$\beta(\bar{\alpha}(t)) = -(\nu_+ - \nu_-)\sin\bar{\alpha}(t)$$

Applications: Quantum graph ①



- Star graph with N edges (The most simplest model of quantum wire junctions)

- Boundary condition

$$(\mathbb{1}-U)\begin{pmatrix}\psi(0_1)\\ \vdots \\ \psi(0_N)\end{pmatrix} - iL_0(\mathbb{1}+U)\begin{pmatrix}\psi'(0_1)\\ \vdots \\ \psi'(0_N)\end{pmatrix} = \vec{0}, \quad U \in U(N)$$

- Exact β-function

$$\beta(\bar{\alpha}_j(t)) = -\sin\bar{\alpha}_j(t), \quad j = 1, \cdots, N$$

(α_j: jth eigenphase of U)

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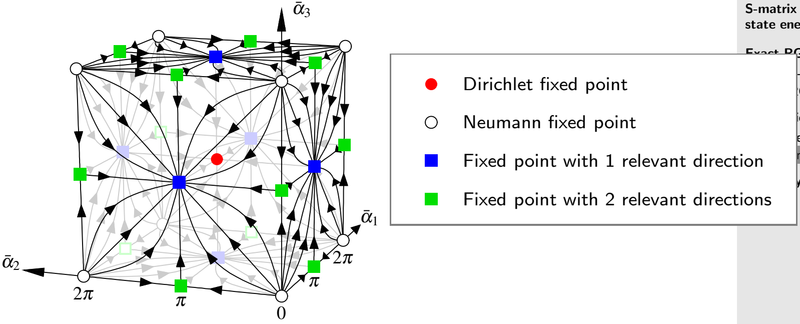
- Exact RG flow of boundary conditions for $N = 3$ (Y-junction)



(Arrows indicate the directions toward IR.)

- Rich phase structure

- There exist $2^3 = 8$ fixed points

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- $U = \mathrm{e}^{i\alpha_+} P_+ + \mathrm{e}^{i\alpha_-} P_- \in U(2)$
- Running eigenphase $\bar{\alpha}_\pm(t) = 2\arctan\left(\mathrm{e}^{-t}\tan\frac{\alpha_\pm}{2}\right)$
- Projection operator $P_\pm$: Exactly marginal
- UV stable fixed point: Neumann boundary condition
- IR stable fixed point: Dirichlet boundary condition