On twisted symmetries and QM with a magnetic field on NC tori

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Introduction

Deformation quantization using Drinfel'd twists $\mathcal{F}$ is a powerful tool to build noncommutative deformations of a space(time) X, of quantum theories on X and of their symmetries.

$$\overline{\mathcal{F}}_t := \exp\left(\frac{i}{2}\partial_a \otimes \theta^{ab}\partial_b\right) \equiv \exp\left(\frac{i}{2}\partial^t \otimes \theta\partial\right), \qquad \theta^{ab} = -\theta^{ba} \in \mathbb{R}$$
$$[f\star' g](x) := \cdot\left[\overline{\mathcal{F}}_t(\triangleright\otimes\triangleright)(f\otimes g)\right] = f(x)\exp\left[\frac{i}{2}\overleftarrow{\partial_a}\,\theta^{ab}\,\overrightarrow{\partial_b}\right]g(x), \tag{1}$$

deforms the pointwise product $\cdot$ of smooth functions f, g both on $X = \mathbb{R}^m$ (Grönewold-Moyal-Weyl $\star$-product) and on $X = \mathbb{T}^m$ (Connes-Rieffel $\star$-product); here $x \in \mathbb{R}^m$, $\partial_a = \frac{\partial}{\partial x^a}$, and for $\mathbb{T}^m$ f, g are meant periodic: $f(x + 2\pi L) = f(x)$, $L \in \mathbb{Z}^m$.
A better definition of $\star'$ (larger domains!) uses the Fourier transforms/series of f, g.
$\partial_a \in \mathbf{g} \equiv$ Lie algebra of the group G of symmetries of X. The twist $\mathcal{F}_t \equiv \overline{\mathcal{F}}_t^{-1} \in U\mathbf{g} \otimes U\mathbf{g}$ determines also the deformation $H \rightsquigarrow \hat{H}$ of the symmetry Hopf algebra $H = U\mathbf{g}$.
As $\mathcal{X} := C^\infty(\mathbb{T}^m)$ is H-module algebra, so is $\widehat{\mathcal{X}} \equiv \mathcal{X}_{\star'}$ a $\hat{H}$-module algebra.

Scalar quantum particle of electric charge q on $X = \mathbb{T}^m$ with a magnetic field $B = dA$:
$\boldsymbol{\psi} \in \Gamma(X, E) \sim e\mathcal{X}^n$.
Here $\Gamma(X, E)$:=space of sections of the associated hermitean line bundle $E \overset{\pi}{\mapsto} X$,
$n \in \mathbb{N}$, $e \in M_n(\mathcal{X})$ is a projector, and used the Serre-Swan theorem.

Standard approach: right $\mathcal{X}$-module structure of $\Gamma(X, E) \sim e\mathcal{X}^n$ also deformed by $\star'$,
$\boldsymbol{\psi} f \rightsquigarrow \boldsymbol{\psi}\star' f$, $f \in \mathcal{X}$.

$\Gamma(X,E) \sim e\mathcal{X}^n$ is not symmetric under translation group $G = \mathbb{T}^m$, nor is $e_{\star'}\mathcal{X}^n_{\star'}$ under $\widehat{U\mathbf{g}}$.
We point out: $\Gamma(X,E)$ is symmetric under a central extension of G, the *projective translation group* G_Q.
$G_Q = \mathbb{T}^k \times$Heisenberg group, electric charge operator $Q \equiv$ central generator. G, G_Q have the same action on $\mathcal{X}$. Interesting result in itself.
Here we deform by a $\mathcal{F} \in U\mathbf{g}_Q \otimes U\mathbf{g}_Q$ and related $\star$ to "preserve" the symmetries.
To find G_Q we describe $\Gamma(X,E)$ as a subspace $\mathcal{X}^V$ of $C^\infty(\mathbb{R}^m)$ characterized by a quasiperiodicity condition, i.e. periodicity up to a phase $V(L,x) \in U(1)$.
This can be used also for physics on $\mathbb{R}^m$.

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2. Mapping $\Gamma(X,E) \xrightarrow{\sim} \mathcal{X}^V$, defining G_Q

$$\mathcal{X}^V := \{\psi \in C^\infty(\mathbb{R}^m) \;\; | \;\; \psi(x+2\pi L) = V(L,x)\,\psi(x) \qquad \forall\, x \in \mathbb{R}^m,\; L \in \mathbb{Z}^m\}, \tag{2}$$

$\mathcal{X}^V$ is well-defined $\Leftrightarrow$

$$\begin{array}{ccc} x & \xrightarrow{L+L'} & x{+}2\pi(L{+}L') \\ & {\scriptstyle L}\searrow \quad \nearrow{\scriptstyle L'} & \\ & x{+}2\pi L & \end{array}$$

is commutative $\Leftrightarrow$

$$V(L{+}L',x) = V(L, x{+}2\pi L')V(L,x). \tag{3}$$

It must be $\nabla_a : \mathcal{X}^V \mapsto \mathcal{X}^V$. $\quad QB_{ab} = \frac{i}{2}[\nabla_a, \nabla_b] : \mathcal{X}^V \mapsto \mathcal{X}^V \;\Rightarrow\; B_{ab}(x)$ periodic:

$$B_{ab}(x) = \frac{1}{2}\beta^A_{ab} + \underbrace{\sum_{L\neq 0} \beta^L_{ab} e^{iL\cdot x}}_{B'_{ab}(x)} \quad \Rightarrow \quad A_a(x) = \frac{1}{2}x^b\beta^A_{ba} + \underbrace{A'_a(x)}_{\text{periodic, } B'=dA'} \tag{4}$$

$$\nabla_a\psi \in \mathcal{X}^V \quad \text{if} \quad V(L,x) = e^{-i\pi L^t\beta^A x}; \qquad \stackrel{(3)}{\Rightarrow} \qquad \pi\beta^A_{ab} \in \mathbb{Z}$$

$$\nabla_a = -i\partial_a + QA_a(x) = p_a + QA'_a(x), \qquad p_a := -i\partial_a + \frac{Q}{2} x^b \beta^A_{ba} \qquad u^a := e^{ix^a}$$

Set $u^L := e^{iL\cdot x}$. $Q, p_a, u^L\cdot : \mathcal{X}^V \mapsto \mathcal{X}^V$; belong to the $*$-algebra of observables $\mathcal{O} \equiv$ algebra of polynomials in $Q, p_1, ..., p_m$ with coefficients f in $\mathcal{X}$, constrained by

$$[p_a, p_b] = -i\beta^A_{ab} Q, \qquad [Q, \cdot] = 0, \qquad [p_a, f] = -i(\partial_a f), \tag{5}$$

Q, p_a generate a real Lie algebra $\mathbf{g}_Q = \mathbb{R}^k \oplus$ Heisenberg, $k \leq m$. Group G_Q consists of

$$\Gamma_{(z^0, z)} = e^{i[Qz^0 + p\cdot z]}; \qquad [\Gamma_{(z^0, z)}\psi](x) = e^{iq\left[z^0 + x^t \frac{\beta^A}{2}\right]}\psi(x+z) \in \mathcal{X}^V. \tag{6}$$

G, G_Q have same actions if $q = 0$. Gauge transformation $U(x)$ = unitary transformation,

$$\mathcal{X}^V \mapsto \mathcal{X}^{V^U}, \qquad \psi \mapsto \psi^U = U\psi, \qquad p_a \mapsto p_a^U = Up_aU^{-1} \qquad u^a \mapsto u^a. \tag{7}$$

Choosing $U(x) = e^{i\frac{q}{4}x^t\beta^S x}$ and setting $\beta := \beta^A + \beta^S$, we find for $\psi \in \mathcal{X}^{V^U} =: \mathcal{X}^\beta$

$$\psi(x + 2\pi L) = e^{-iq\pi L^t\beta(x+L\pi)}\psi(x), \qquad p_a = -i\partial_a + x^b\beta_{ba}\frac{Q}{2}. \tag{8}$$

Let: P be the canonical cover map $P : x \in \mathbb{R}^m \mapsto u \in \mathbb{T}^m \sim \mathbb{R}^m/\mathbb{Z}^m$, $\{X_i\}$ a finite open cover of $\mathbb{T}^m$, $\forall i\; W_i \subset \mathbb{R}^m$ such that $P_i \equiv P|_{W_i} : W_i \mapsto X_i$ is invertible. Let

$$\psi_i(u) := \psi[P_i^{-1}(u)], \qquad A_{ia}(u) := A_a[P_i^{-1}(u)], \qquad U_i(u) := U[P_i^{-1}(u)], \qquad u \in X_i,$$

$$\overset{(3)}{\Rightarrow} \qquad \psi_i = c_{ij}\psi_j \quad \text{in } X_i \cap X_j, \qquad c_{ik} = c_{ij}c_{jk} \quad \text{in } X_i \cap X_j \cap X_k \tag{9}$$

Therefore $\{(X_i, \psi_i, A_i)\}$ defines a trivialization of a section of a line bundle $E \overset{\pi}{\mapsto} X$ with connection A of Chern numbers $n_{ab} := \pi\beta^A_{ab} = \frac{1}{2\pi}\phi_{ab} \in \mathbb{Z}$.
$\{(X_i, U_i)\}$ defines a gauge transformation; if $U(x)$ is periodic (global gauge tr.) $c^U_{ij} = c_{ij}$.
$c_{ij}(u) = \exp\left\{-i\frac{q}{2}[P_i^{-1}(u)]^t\beta^A[P_j^{-1}(u)]\right\}$ in the gauge $\mathcal{X}^{\beta^A}$

This map $[\mathcal{X}^V] \mapsto \Gamma(X, E)$ can be inverted, so we can identify $[\mathcal{X}^V] \simeq \Gamma(X, E)$.

$$(\psi', \psi) := \int_X \psi'^*\psi, \qquad \int_X := \int_{\mathbb{T}^m} d^m x \tag{10}$$

defines an Hermitean structure as $\psi'^*\psi$ is periodic; p_a, ∇_a are essentially self-adjoint.
We shall call $\mathcal{H}^V$ the Hilbert space completion of $\mathcal{X}^V$.

Fixed a $\psi_0 \in \mathcal{X}^V$ vanishing nowhere, $\psi\psi_0^{-1}$ is well-defined and periodic, i.e. in $\mathcal{X}$, whence the decomposition $\mathcal{X}^V = \mathcal{X}\,\psi_0$: ψ_0 is cyclic and separating.

3.1 Twisting $H = U\mathbf{g}$ to a noncocomm. Hopf algebra $\hat{H}$

If not familiar with Hopf algebras: start with cocommutative Hopf algebra $U\mathbf{g}$: then

$$\varepsilon(\mathbf{1}) = 1, \qquad \Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}, \qquad S(\mathbf{1}) = \mathbf{1},$$

$$\varepsilon(g) = 0, \qquad \Delta(g) = g\otimes\mathbf{1} + \mathbf{1}\otimes g, \qquad S(g) = -g, \qquad \text{if } g \in \mathbf{g};$$

ε, Δ are extended to all of $H = U\mathbf{g}$ as $*$-algebra maps, S as a $*$-antialgebra map:

$$\begin{aligned} &\varepsilon : H \to \mathbb{C}, && \varepsilon(ab) = \varepsilon(a)\varepsilon(b), && \varepsilon(a^*) = [\varepsilon(a)]^*, \\ &\Delta : H \to H\otimes H, && \Delta(ab) = \Delta(a)\Delta(b), && \Delta(a^*) = [\Delta(a)]^{*\otimes *}, \\ &S : H \to H, && S(ab) = S(b)S(a), && S\{[S(a^*)]^*\} = a. \end{aligned} \tag{11}$$

The extension of Δ is unambiguous, as $\Delta\big([g, g']\big) = \big[\Delta(g), \Delta(g')\big]$ if $g, g' \in \mathbf{g}$.

ε gives the trivial representation, Δ, S are the abstract operations by which one constructs the tensor product of any two representations and the contragredient of any representation, respectively; S is uniquely determined by Δ.

Real deformation parameter λ. $\hat{H}, H[[\lambda]]$ have

1. same $*$-algebra (over $\mathbb{C}[[\lambda]]$) and counit ε
2. coproducts $\Delta, \hat{\Delta}$ related by

$$\Delta(g) \equiv \sum_I g^I_{(1)} \otimes g^I_{(2)} \quad \longrightarrow \quad \hat{\Delta}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1} \equiv \sum_I g^I_{(\hat{1})} \otimes g^I_{(\hat{2})}$$

3. antipodes $S, \hat{S}$ s.t. $\hat{S}(g) = \beta S(g)\beta^{-1}$, with $\beta = \sum_I \mathcal{F}^{(1)}_I S\left(\mathcal{F}^{(2)}_I\right)$.

where the *twist* [Drinfel'd 83] is for our purposes a unitary element $\mathcal{F} \in (H \otimes H)[[\lambda]]$ fulfilling

$$\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + O(\lambda), \qquad (\epsilon \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \epsilon)\mathcal{F} = \mathbf{1},$$

$$(\mathcal{F} \otimes \mathbf{1})[(\Delta \otimes \mathrm{id})(\mathcal{F})] = (\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)(\mathcal{F})] =: \mathcal{F}_3. \qquad (12)$$

$\hat{H}$ has unitary triangular structure $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$.

Here $H = U\mathbf{g}_Q$, $\mathcal{F} \in (U\mathbf{g}_Q \otimes U\mathbf{g}_Q)[[\lambda]]$; here for simplicity only Reshetikhin twists:

$$\mathcal{F} = e^{\frac{i}{2}(p^t \otimes \theta p + \mu \cdot p \wedge Q)}, \qquad \theta = \tilde{h}^t \Theta \tilde{h}, \qquad \text{where } \tilde{h} \text{ solves } \tilde{h}\beta^A \tilde{h}^t = 0 \qquad (13)$$

($\theta^{ab} = \lambda \vartheta^{ab}$, $\mu^a = \lambda \nu^a$; the term $\mu \cdot p \wedge Q$ irrelevant for 1-particle system).
(13) implies $\mathrm{tr}(\theta\beta^A) = 0$, $\theta\beta^A\theta = 0$.

Such $\theta \neq 0$ exist only if $m \geq 3$: Simple nontrivial deformations are $m = 3$ with

$$\beta^A = \begin{pmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & -\eta & 0 \end{pmatrix}, \qquad \Rightarrow$$

$$\mathcal{F} = e^{\frac{i}{2}\eta\, p_2 \wedge p_3}, \qquad \hat{\Delta}(Q) = \Delta(Q), \qquad \hat{\Delta}(p_a) = \Delta(p_a) + \delta_{a1}\frac{\eta b}{2}\, p_3 \wedge Q, \tag{14}$$

and $m = 4$ with

$$\beta^A = \begin{pmatrix} 0 & -b & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & -c \\ 0 & 0 & c & 0 \end{pmatrix}, \qquad \theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & -\eta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow$$

$$\mathcal{F} = e^{\frac{i}{2}\eta\, p_2 \wedge p_3}, \qquad \hat{\Delta}(Q) = \Delta(Q), \qquad \hat{\Delta}(p_a) = \Delta(p_a) + \delta_a^1 \frac{\eta b}{2}\, p_3 \wedge Q + \delta_a^4 \frac{\eta c}{2}\, p_2 \wedge Q. \tag{15}$$

3.2 Twisting H-module ($*$-)algebras

Let $\mathcal{A}$ be a H-module ($*$)-algebra (over $\mathbb{C}$), $V(\mathcal{A})$ the vector space underlying $\mathcal{A}$. $V(\mathcal{A})[[\lambda]]$ gets a $\hat{H}$-module ($*$-)algebra $\mathcal{A}_\star$ when endowed with the product (and $*$-structure)

$$a \star a' := \sum_I \left(\overline{\mathcal{F}}_I^{(1)} \triangleright a\right) \left(\overline{\mathcal{F}}_I^{(2)} \triangleright a'\right), \qquad (a^{\hat{*}} := S(\beta) \triangleright a^*). \tag{16}$$

In fact, $\star$ is associative by (2), fulfills $(a\star a')^{\hat{*}} = a'^{\hat{*}} \star a^{\hat{*}}$ and

$$g \triangleright (a\star a') = \sum_I \left[g^I_{(\hat{1})} \triangleright a\right] \star \left[g^I_{(\hat{2})} \triangleright a'\right]. \tag{17}$$

A left H-equivariant ($*$-)$\mathcal{A}$-bimodule $\mathcal{M}$ is deformed into a left $\hat{H}$-equivariant $\mathcal{A}_\star$-$*$-bimodule $\mathcal{M}_\star$ w.r.t. the left, right $\mathcal{A}_\star$-multiplication: (16) for all $a \in \mathcal{A}_\star$, $a' \in \mathcal{M}_\star$ and $a \in \mathcal{M}_\star$, $a' \in \mathcal{A}_\star$. **If $\mathcal{A}$ defined by generators a_i and relations**, **then also $\mathcal{A}_\star$ is**, with same Poincaré-BirkhoffWitt series. *Generalized Weyl map* $\wedge : f \in \mathcal{A} \to \hat{f} \in \mathcal{A}_\star$ defined by the equations

$$f(a_1, a_2, ...)\star = \hat{f}(a_1\star, a_2\star, ...) \qquad \text{in } V(\mathcal{A}) = V(\mathcal{A}_\star) \tag{18}$$

If $\exists$ a ($*$)-algebra map $\sigma : H \mapsto \mathcal{A}$ such that $g \triangleright a = \sum_I \sigma\left(g^I_{(1)}\right) a\, \sigma\left(Sg^I_{(2)}\right)$, then $\exists$ a $\hat{H}$**-module $*$-algebra isomorphism** $D^\sigma_{\mathcal{F}} : \mathcal{A}_\star \leftrightarrow \mathcal{A}[[\lambda]]$ (*deforming map*) defined by

$$D^\sigma_{\mathcal{F}}(a) := \sum_I \left(\overline{\mathcal{F}}_I^{(1)} \triangleright a\right) \sigma\left(\overline{\mathcal{F}}_I^{(2)}\right) \tag{19}$$

Change notation: $a_i \star a_j \rightsquigarrow \hat{a}_i \hat{a}_j$, $\hat{f}(a_i\star) \rightsquigarrow \hat{f}(\hat{a}_i)$, $\mathcal{A}_\star \rightsquigarrow \widehat{\mathcal{A}}$, $\mathcal{M}_\star \rightsquigarrow \widehat{\mathcal{M}}$,...

Twisting $\mathcal{X}, \mathcal{X}^V, \mathcal{O}, \ldots$

Weyl form of the comm. rel. $[Q, x^a] = [Q, p_a] = 0$, $[p_a, x^b] = -i\delta^b_a$, $[p_a, p_b] = -i\beta^A_{ab}Q$:

$$e^{i(h\cdot x+p\cdot y+\frac{Q}{2}y^0)}e^{i(k\cdot x+p\cdot z+\frac{Q}{2}z^0)} = e^{i\big[(h+k)\cdot x+p\cdot(y+z)+\frac{Q}{2}(y^0+z^0)\big]}\, e^{-\frac{i}{2}[k\cdot y-h\cdot z-Qy\beta^A} \tag{20}$$

for any $h, k, y, z \in \mathbb{R}^m$ and $y^0, z^0 \in \mathbb{R}$.

$$Y := \{e^{i(Qh\cdot x+p\cdot z+\frac{Q}{2}z^0+is)} \mid h, z \in \mathbb{R}^m,\ \ z^0, s \in \mathbb{R}\} \simeq \text{Heisenberg Group} \supseteq G,$$

$$\mathcal{C} := \text{the universal } C^*\text{-algebra generated by the } y \in Y. \tag{21}$$

The group law of Y can be read off (20) replacing $h \to Qh$. Fixed a twist (13) and applying the deformation procedure to $\mathcal{C}$ one obtains $\widehat{\mathcal{C}} \sim \mathcal{C}$ with basic commutation relations

$$e^{i(Qh\cdot\hat{x}+\hat{p}\cdot y+\frac{Q}{2}z^0)}e^{i(k\cdot\hat{x}+\hat{p}\cdot z+\frac{Q}{2}z^0)} = e^{i[(h+k)\cdot\hat{x}+\hat{p}\cdot(y+z)]-\frac{i}{2}\left[k\cdot y-h\cdot z+(h+\hat{Q}\beta^A y)^t\theta(k+\hat{Q}\beta^A z)\right]} \tag{22}$$

for all $h, k, y, z \in \mathbb{R}^m$; for $y = z = 0$ the second becomes exactly as on Moyal space

$$e^{ih\cdot\hat{x}}e^{ik\cdot\hat{x}} = e^{i(h+k)\cdot\hat{x}}e^{-\frac{i}{2}h\theta k}. \tag{23}$$

$$[\hat{Q},\hat{f}]=0, \qquad [\hat{p}_a, e^{ik\cdot\hat{x}}]=e^{ik\cdot\hat{x}}[k+\hat{Q}k\theta\beta^A]_a, \qquad [e^{ik\cdot\hat{x}},\hat{x}^a]=e^{ik\cdot\hat{x}}(\theta k)^a,$$
$$\hat{Q}^{\hat{*}}=\hat{Q}, \qquad \left[e^{i(k\cdot\hat{x}+\hat{p}\cdot y)}\right]^{\hat{*}}=e^{-i(k\cdot\hat{x}+\hat{p}\cdot y)}, \tag{24}$$

Given β^A,θ fulfilling (4), (13), we start with a gauge where (the symmetric part of) β fulfills

$$\beta\theta\equiv(\beta^A+\beta^S)\theta=0 \tag{25}$$

(such a β always exists). In such a gauge we define $\widehat{\mathcal{X}}^\beta\subset\widehat{\mathcal{S}}'$ as the spaces (and $\hat{H}$-$*$-modules) of objects of the form

$$\hat{\psi}(\hat{x})=\int_{\mathbb{R}^m}d^mk\ e^{ik\cdot\hat{x}}\tilde{\psi}(k) \tag{26}$$

with $\tilde{\psi}(k)$ fulfilling

$$\int_{\mathbb{R}^m}d^mk\,|\tilde{\psi}(k)|\big(1+|k|\big)^h<\infty, \qquad \tilde{\psi}(k+\pi qL^t\beta)=\ e^{i\pi L\cdot[2k+q\pi\beta L]}\tilde{\psi}(k), \tag{27}$$

for all $h=0,1,2,...$ and $L\in\mathbb{Z}^m$. This ensures the noncommutative quasiperiodicity property

$$\hat{\psi}(\hat{x}+2\pi L)=e^{-iq\pi L^t\beta(\hat{x}+L\pi)}\,\psi(\hat{x}), \qquad L\in\mathbb{Z}^m, \tag{28}$$

completely analogous to (8).

$$-i\hat{D}_a = -i\hat{\partial}_a + \hat{A}_a\hat{Q} = \hat{p}_a + \hat{A}'_a\hat{Q}, \qquad \hat{p}_a = -i\hat{\partial}_a + \hat{x}^b\beta_{ba}\frac{\hat{Q}}{2}, \qquad \hat{A}'_a \in \widehat{\mathcal{X}}, \tag{29}$$

$\widehat{\mathcal{X}}^\beta$ is mapped into itself by multiplication by any $\hat{u}^L$, the action of $\hat{p}_a$. Moreover $\hat{\psi}^{\hat{*}}\hat{\psi}' \in \widehat{\mathcal{X}}$ for any $\hat{\psi}, \hat{\psi}' \in \widehat{\mathcal{X}}^\beta$; we define a Hermitean structure on $\widehat{\mathcal{X}}^\beta$ by

$$(\hat{\psi}, \hat{\psi}') := \int_{\hat{X}} \hat{\psi}^{\hat{*}}\hat{\psi}' = \int_{\mathbb{T}^n} d^n x\, \psi^*\psi' = (\psi, \psi'); \tag{30}$$

where $\int_{\hat{X}}$ is Connes-Rieffel integration on the noncommutative torus. By the last equalities there exists a Hilbert space isomorphism $\widehat{\mathcal{H}}^\beta \simeq \mathcal{H}^\beta$, where $\widehat{\mathcal{H}}^\beta$ is the Hilbert space completion of $\widehat{\mathcal{X}}^\beta$. $\hat{p}_a, \hat{\nabla}_a$ are essentially self-adjoint.

Going to a more general gauge is under study...