How unique is COVARIANT STAR PRODUCT?

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Contents:

- Non-uniqueness of the star product
- Covariant resolution of the non-uniqueness problem
- Implications for NC gravity (if time will allow)

Deformation quantization

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Milestones: Bayen, Flato, Frønsdal, Lichnerowicz, Sternheimer (1977).
Fedosov (1986-1990); Kontsevich (1997).
Main idea: replace the algebra A of (smooth) functions on the space-time manifold by the algebra $A[[h]]$ of *formal power series*.
For $a \in A[[h]]$

$$a = a_0 + ha_1 + h^2 a_2 + \dots$$
$$a \star b = a \cdot b + hC_1(a, b) + h^2 C_2(a, b) + \dots$$

Summing up the series does not make any sense in this approach.

Associativity $a \star (b \star c) = (a \star b) \star c$ yields that $C(a,b) - C(b,a)$ is a Poisson structure. On functions,

$$C(a,b) - C(b,a) = \omega^{\mu\nu} \partial_\mu a \cdot \partial_\nu b$$

Main problem of deformation quantization: given a Poisson structure find all formal deformations along this structure.
Difficulties:
– The (Konstsevich) procedure is not covariant.
– There are *too many* star products corresponding to a given Poisson structure: if $\star$ is an admissible product, any product of the form

$$a \star' b = D^{-1}((Da) \star (Db))$$

is also an admissible product, where

$$D = 1 + hL$$

with a formal differential operator

$$L = \sum_{k=1}^{\infty} L^{\mu_1 \cdots \mu_k}(h, x) \nabla_{\mu_1} \dots \nabla_{\mu_k}$$

Therefore, the star product depends on an infinite number of fields!
Note: the algebras corresponding to different D are isomorphic, but the field theories based on two such products are not equivalent (for a field theory actions we need derivatives and traces).
Before consutructing a physical theory on a noncommutative space one must reduce this (enormous!) ambiguity.

Covariant star product

on a symplectic manifold can be constructed by extending to all tensors the covariantization of the Moyal product done by BFFLS. Let TM be a tangent bundle, and T^*M be a cotangent bundle. Let $\alpha_{n,m}$ be a tensor field, $\alpha_{n,m} \in TM^n \otimes T^*M^m \equiv T^{n,m}$. This means, $\alpha_{n,m}$ has n contravariant and m covariant indices. Let us choose a Christoffel symbol on M such that the symplectic form is covariantly constant,

$$\nabla_\mu \omega_{\nu\rho} = \partial_\mu \omega_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\omega_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\omega_{\nu\sigma} = 0.$$

Therefore, M becomes a Fedosov manifold. Let us suppose that this connection is flat and torsion-free, i.e.,

$$[\nabla_\mu, \nabla_\nu] = 0.$$

Locally, one can choose a coordinate system such that $\omega^{\mu\nu} = const.$ and $\Gamma^\sigma_{\mu\nu} = 0$ (the Darboux coordinates).

Define the star product by

$$\alpha \star \beta = \sum_k \frac{h^k}{k!} \omega^{\mu_1 \nu_1} \dots \omega^{\mu_k \nu_k} \times$$
$$\times (\nabla_{\mu_1} \dots \nabla_{\mu_k} \alpha) \cdot (\nabla_{\nu_1} \dots \nabla_{\nu_k} \beta) \, .$$

Properties:
1. Covariance: $T^{n_1 m_1} \star T^{n_2 m_2} \subset T^{n_1+n_2, m_1+m_2}$.
2. Associativity $\alpha \star (\beta \star \gamma) = (\alpha \star \beta) \star \gamma$.
3. Stability on covariantly constant tensors: $\alpha \star \beta = \alpha \cdot \beta$ if $\nabla \alpha = 0$ or $\nabla \beta = 0$.
4. Derivation:

$$\nabla \alpha \star \beta = (\nabla \alpha) \star \beta + \alpha \star (\nabla \beta).$$

Let us impose these conditions on a generic star product.

Main Theorem

: Let M be a symplectic manifold with a symplectic structure $\omega^{\mu\nu}$ and a flat torsionless connection ∇. Any covariant star product on the space of tensor fields over M satisfying 1-4 can be represented as

$$\alpha \star_N \beta = D^{-1}(D\alpha \star_R D\beta)\,, \tag{1}$$

where the product $\star_R$ is the covariantized Moyal star product with a renormalized ω:

$$\omega_R^{\mu\nu} = \omega^{\mu\nu} + h^2 \omega_1^{\mu\nu} + \dots$$

(all correction terms $\omega_j^{\mu\nu}$ are covariantly constant), $D = 1 + hL$,

$$L = \sum_{k=1}^{\infty} L^{\mu_1 \cdots \mu_k}(x) \nabla_{\mu_1} \dots \nabla_{\mu_k}$$

and the coefficients $L^{\mu_1 \cdots \mu_n}$ in the gauge operator D are covariantly constant and contain odd powers of the deformations parameter h. The coefficient with n=1 vanishes.

Proof: brute force.

Comments:

Constant renormalizations of ω are harmless, since ω_R may be considered as a "physical" value of noncommutativity.

The Theorem reduces the ambiguity in the star product from infinite functional dimension to just a few fields.

What is the physical meaning of the "gauge" freedom remaining in L?

Integration

On symplectic manifolds there is a natural diffeomorphism invariant integration measure $d\mu(x) = (\det(\omega^{\mu\nu}))^{-1/2}$.

Let us consider a classical action

$$S = \int d\mu(x) P(f_i, \nabla)_{\star_N} \,,$$

where P is a diffeomorphism invariant polynomial, f_i are some fields. One can easily show

$$S = \int d\mu(x) D^{-1}(P(Df_i, \nabla)_{\star_R}) = \int d\mu(x) P(Df_i, \nabla)_{\star_R} \,.$$

This means, that the replacement $\star_N$ by $\star_R$ is compensated by the transformation $f_i \to Df_i$. Since the operator D is invertible, the theories based on the two star products are classically equivalent.

Conclusion

: A natural covariant star product on a symplectic manifold is essentially uniquely defined by a symplectic structure $\omega^{\mu\nu}$ and a flat torsionless symplectic connection. The remaining freedom is physically insignificant.
One can, therefore, construct noncommutative theories by using the same ideology as in, e.g., general relativity.
Done: some 2D models.
An independent dynamical principle for ω and ∇ is needed.

Further problems

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- Extension to arbitrary Poisson manifolds (regular PM - easy, generic - very hard).
- Transition to strict deformation quantization (convergence in h).
- Full-scale noncommutative geometry.

Diffeomorphisms in NC gravity

Main statement: Without a relation between the Poisson (symplectic) structure and the Riemann structure any NC gravity has zero predictive power.

Example: We are in 2D. Let the complex zweiben be diff. equivalent to the flat one. Let the strar product be diff. equivalent to Moyal. The metric can be anything since the "trivialization" occurs in different coordinate systems. [There is a 2D NC gravity behind].

Let us take a coordinate system where $\star = \star_M$. Flat zweiben reads

$$e_\mu = u \star \nabla_\mu E, \qquad \bar{e}_\mu = \nabla_\mu \bar{E} \star u^{-1}$$

u is a $U(1)_\star$ Euclidean Lorentz, E is a complex scalar describing diffeos.

Let us take $E = \sin(x^1) + i \sin(x^2)$ (some simple form) and calculate the metric

$$g_{\mu\nu} = \frac{1}{2}(\bar{e}_\mu \star e_\nu + \bar{e}_\nu \star e_\mu)$$

which reads

$$g_{\mu\nu} = \begin{pmatrix} \cos^2 x^1 & -\sin\theta \sin x^1 \sin x^2 \\ -\sin\theta \sin x^1 \sin x^2 & \cos^2 x^2 \end{pmatrix}.$$

Small x^1, x^2 – this is a sphere or a hyperbolic space depending on the sign of θ.
At finite x^1, x^2 the signature changes to the pseudoeuclidean.
Absolutely wild behavior!